and the contribution made by mathematics in the prosecution of the war. The interest evoked by the papers on the joint program showed that a further half day might have been profitably devoted to these papers and their discussions.

The program, accompanied by numbered abstracts is grouped in two parts. Professor E. R. Hedrick was chairman of the program committee.

I. Conference on War Time Experiences.

"Deductions from War Time Experiences with respect to the Teaching of Mathematics." A conference participated in by representatives of various colleges and universities in which Students' Army Training Corps were located, including (1) H. E. Slaught, University of Chicago; (2) R. P. Baker, University of Iowa; (3) W. D. Cairns, Oberlin College; (4) A. R. Crathorne, University of Illinois; (5) D. R. Curtiss, Northwestern University; (6) W. B. Ford, University of Michigan; (7) A. M. Kenyon, Purdue University.

(8) "The Ensign School," Professors E. J. Moulton and R. E. Wilson, Northwestern University.

(9) "An Experiment in Supervised Study," Professor D. R. Curtiss, Northwestern University.

(10) Discussion by Professor H. L. Rietz, University of Iowa.

(1) Professor Slaught urged the importance of our deriving in this conference the greatest gain for the future from our recent experiences. He stated as one deduction from the experiences with the war classes the need of more concrete and practical applications and less of the complicated and abstract manipulation in all elementary courses. In this connection he called attention to the need of revising the definitions of the preparatory "units" and formulated the following statement on this subject:

"In view of the fact that there is a widespread desire among the secondary teachers of mathematics, as well as among many college teachers, for a reconsideration of the definitions of the 'Units' in preparatory courses in mathematics as formulated by a committee of the American Mathematical Society in 1903 and used as the basis of the examination questions set by the College Entrance Examination Board and by the Regents of the State of New York, and, further, in view of the fact that the Mathematical Association of America has a standing committee on Mathematical Requirements consisting of Professor J. W. Young, chairman, Professor D. E. Smith, Professor E. H. Moore, Professor C. N. Moore, Professor A. R. Crathorne, and Professor H. W. Tyler, all of whom are members of both the Society and the Association, which committee has been enlarged by associating with it a representative from each of the large secondary associations in New England, the Middle States, and the Middle Western States, thus making it truly representative and national in character, it would seem to be most appropriate for the Council of the Society and the Council of the Association to join in formally referring the whole matter to this National Committee for consideration." A resolution embodying this action was later passed by the Councils of both organizations.

(2) Professor Baker reported upon the teaching of trigonometry under the S. A. T. C. He pointed out specific instances where the necessary shortening of the course resulted in the students filling the gaps in the theory outside of class; this raised the question of the wisdom of such abbreviation. The net result seemed to be that the capable students overcame the added difficulties but that the total number acquiring the subject satisfactorily was considerably reduced.

(3) Professor Cairns described a course in map sketching and map reading which he supervised in the S. A. T. C. in Oberlin College, fulfilling the requirements called for by the Committee on Education and Special Training. Somewhat more than one hundred twenty-year-old men were given familiarity and practice in the field methods of pacing, obtaining bearing and azimuth, plane table mapping, profile leveling, measuring angles with transit, measurement of slopes, sketching and reading topographic maps, with a modicum of trigonometry. Working with men of that age who had, in large part, had no college training after graduation from the high school, one obtained a new and evidently valuable revision of one's previous notion of what constituted mathematics, and the necessity of joining the students' college mathematics and his preceding courses more skillfully, was impressed upon one as never before.

(4) Professor Crathorne reported that at the University of Illinois some 2100 students were registered in trigonometry which was given in two courses, one three hours a week for the non-engineering S. A. T. C. students and one four hours a week for the S. A. T. C. students registered in the College of Engineering. The extra hour in the latter course was devoted to the slide rule and the applications of trigonometry to military and nautical problems, given to the student in sheet and pamphlet form. The military authorities arranged the supervised study; this was not wholly a success due to the discomforts and noise in the barracks. The conflicts of military duties with academic studies together with the influenza epidemic caused a great many cuts from classes. Many students were absent for weeks at a time and naturally became discouraged. To help out in this situation a week's review was given about the middle of the term and much private help afforded to students asking it.

Changes in the University of Illinois mathematical curriculum due to war experience will doubtless be slight, perhaps more in the attitude of the teacher toward practical applications than in anything else. There seemed to be a general feeling among the instructors that the applied side of mathematical teaching had at least not been over-emphasized. The teaching force was augmented by instructors from other departments particularly from those in the college of engineering and one result of the experience will be a better understanding of the problem of teaching mathematics to freshmen by men who will have these students later in their engineering course.

At present it is not expected that there will be any marked change in the character of advanced courses in mathematics due to the military experience. Some advanced courses in special topics of applied mathematics like "Ballistics" and "Aërodynamics" may be given by the mathematics department or some

other department. There has grown up in other departments of the University a better feeling towards the teaching of mathematics and a respect for its usefulness, for example, several departments requiring mathematics for prerequisites have asked for more mathematics, whereas formerly there was a tendency to hold the mathematics to a minimum.

While not concerned with mathematics, a very important result of our war experience is the continuation of our "war aims" course into a course of training for citizenship to be open to the students of all the colleges of the university.

(5) Professor Curtiss stated that at Northwestern University little attempt had been made to change the usual subject matter of mathematical courses, but that new methods had been tried. The chief innovation he reserved for discussion in his subsequent paper. Seven different freshman courses (including two combining Trigonometry and Navigation) were given, and students who could not carry a course were put back at any time that seemed advisable into the next lower course. Thus every man was kept in some course for which he was presumably fitted, and the fluctuating character of the attendance was partially offset. A number of men had not had a complete course in plane geometry, but had finished a year and a half of algebra. It was noted that these men made better average records in trigonometry than those who presented a year of geometry and a year of algebra.

(6) Professor Ford pointed out that there is a slow but ever-increasing tendency in both college and secondary education to replace mathematics and the other so-called disciplinary subjects by subjects alleged to possess the same disciplinary value yet having the added merit of being closer to the world of affairs and hence of being more "practical." Among the colleges, for example, it is not difficult to find the teacher of economics who will argue that his subject is fully as logical, being founded upon definite laws, as is mathematics, and hence that it may serve equally well to impart disciplinary values, while withal the content of economics bears immediately upon all human affairs. As regards secondary education, there is little or no scruple against even replacing the older disciplinary studies by those having no such value at all, as is instanced by the common and increasing practice of introducing "vocational training" into the public schools. These tendencies, which were already well defined before the war began, will doubtless be intensified in the period lying directly ahead and hence it is an opportune moment for teachers of the older subjects, especially mathematics, to question what their attitude in the matter may well be; in particular, are we to regard the situation as based upon a sound educational policy, notwithstanding it may affect adversely our own interests? The answer is to be found in the often repeated but equally often forgotten principle of psychology which declares that in youth the mind is open and indeed eager to receive idealistic rather than material truth, and that in so far as the individual is deprived of what is thus natural to him at this period his later life is in corresponding measure thrown out of balance. The central duty of education is, or should be, to furnish a background upon which the later routine existence may at least be made tolerable, and, whatever be the

tendencies in vogue, it is a fundamental error to focus the mind prematurely upon matters essentially mundane.

(7) At Purdue University the courses in mathematics given to S. A. T. C. students were 1) Plane trigonometry, 5 hours, including the solution of right and oblique triangles by the usual methods, viz., the laws of sines, cosines, and tangents, and the half-angle formulas, emphasis being put upon computation with and without logarithms; 2) Differential calculus, 3 hours, treating the elements of differentiation and applications to mechanics.

While there were many interruptions caused by sickness and military interference, the effort was made to keep in mind that the primary purpose of the organization was to make soldiers, and in spite of difficulties the results were fairly satisfactory.

The main questions which have been raised by the experience are: (*a*) What topics, if any, in mathematics courses can be omitted without serious loss to technical students? (*b*) Which is the best method: to teach fundamental general methods with drill exercises and then go to the applications, or to begin with applications and treat methods as the need for their use arises? (*c*) Is supervised study desirable under ordinary academic conditions?

(8) In May, 1918, a special course was offered at Northwestern University to students of that institution who had enlisted in the U. S. Naval Auxiliary Force at the Municipal Pier, Chicago. This course consisted of instruction in mathematics, navigation, signalling, infantry drill, swimming, and a study of the *Blue Jacket's Manual*. Eight hours daily was devoted to the above subjects as follows: Mathematics, two hours; navigation, two hours; signalling, two hours; swimming and infantry drill combined, two hours. Selected chapters in the *Blue Jacket's Manual* were assigned to be read and examinations were held to insure that this reading was carefully done. This course, four weeks in length, was continued from May until December 20, a new course beginning every two weeks. All of the work of the course, except signalling, was handled by members of the university faculty. The Navy provided instructors in signalling. The instruction in each subject consisted of lectures approximately one hour in length and supervised study for an additional hour under the supervision of the instructor, who assigned problems and exercises based on the lecture.

Classes as large as 150 students were handled in this manner with satisfactory results. The content of the course in mathematics comprised a short review of algebra, particularly the solution of simple equations and theory of exponents; plane trigonometry, especially the solution of right triangles; logarithms; use of traverse tables and haversine tables of Bowditch; spherical trigonometry, particularly the solution of oblique spherical triangles.

Approximately one thousand took this course. About 90 per cent. of the men had completed plane trigonometry before they entered the course, and the work in mathematics was, for many, merely a review. Practically all the men carried the course satisfactorily. This may be explained on the ground that the men were eager to make as good a showing as possible in order to win promotion

in the Navy in the shortest time possible. The men in the course were detailed to Professor Wilson, who had charge of the course at Northwestern University; and since the course was almost entirely scholastic rather than military almost none of the difficulties of the S. A. T. C. arose. The course proved that fairly satisfactory results in mathematics may be obtained in classes as large as 150 students if there is a strong incentive on the part of the students to master the subject, and if the instructor has sufficient physical strength and nervous energy. It is, however, a task that no one would care to undertake except in time of crisis such as our country was facing during the past year.

(9) The scheme of supervised study followed at Northwestern University and described by Professor Curtiss involved class sessions lasting two and one half clock hours with one ten-minute intermission. The student had one such period each morning, afternoon, and evening, five times a week, and thus carried three subjects, of which one was mathematics. In general fifty minutes were devoted to lecture or recitation, and the rest of the period to study. The same instructor was with the class during the whole two and one half hours. Through excessive absences, lack of interest as demobilization approached, and other causes, this scheme could hardly be said to have had a fair trial. Instructors who tried to carry two large classes found themselves seriously overworked. In spite of these drawbacks, surprisingly good results were obtained. Professor Curtiss discussed the advantages of this method under ordinary academic conditions. He believed that, with some modifications, it should have further trial.

(10) Professor Rietz spoke as follows: The association is certainly indebted to Professor Curtiss for bringing before us the methods and results of this experiment. It is particularly interesting to me that, although supervised study was conducted very differently at the University of Iowa from that at Northwestern, still our experiments left very similar impressions in regard to the possibility of improving our teaching of freshman and sophomore mathematics by a certain amount of supervision of study. Perhaps I should explain that at the University of Iowa the liberal arts students had three large study centers at each of which there were instructors from various departments. There was no assignment of a particular subject to be studied at a particular time. On the other hand, the engineering freshmen and sophomore students were assigned to study mathematics from 7:30 to 9:30 p.m. daily. It was a matter of general comment that the liberal arts students called for more assistance in mathematics than in other subjects. One instructor per hundred students was kept very busy answering questions. I wish to emphasize the point made by Professor Curtiss that through obtaining the answer to some minor question the student was sometimes enabled to make progress and finish the preparation of the lesson, whereas otherwise he would have been entirely unprepared.

The fundamental question that arises in my mind from this experience can be stated as follows: Is it a wise division of time in freshman and sophomore mathematics to make provision for one hour of work by the student in the presence of the teacher to correspond to two hours of outside work? If we could afford to

give one hour to class work of the usual kind and a half hour to consultation and supervision of study, and leave the student one and one half hours for outside work, it seems to me that we would get much better results.

II. JOINT SESSION OF THE ASSOCIATION WITH THE SOCIETY.

(1) "Some Mathematical Features of Ballistics," CAPTAIN A. A. BENNETT (University of Texas), Ord., Washington, D. C.

(2) "How the Map Problem was met in the War," PROFESSOR KURT LAVES, University of Chicago.

(3) "Notes concerning Recent Books on Navigation," ALICE BACHE GOULD, University of Chicago.

(4) "Statistical Methods in Preparation for Service in Statistical Sections of the War Department," PROFESSOR H. L. RIETZ, University of Iowa.

(5) "Ordnance Problems," MAJOR W. D. MACMILLAN (University of Chicago), Ord., Washington, D. C.

(6) "Practical Exterior Ballistics," LIEUTENANT P. L. ALGER, Ord., Aberdeen Proving Ground, Md.

(7) "The Effect of the Earth's Rotation and Curvature on the Path of a Projectile," PROFESSOR W. H. ROEVER (Washington University), Ord., Aberdeen Proving Ground, Md.

(8) "On Low Velocity High Angle Fire," PROFESSOR H. F. BLICHFELDT (Stanford University), Ord., Aberdeen Proving Ground, Md.

(1) Captain Bennett spoke in a more or less informal manner on the subject of projectiles, pointing out some of the most obvious physical and mathematical problems suggested by them. During the passage of the projectile through the bore of the rifle, mechanical questions concerning the uniform or variable pitch of the rifling are presented. On emerging from the muzzle problems of nutation and precession arise at once. The question of air resistance presents numerous meteorological problems, and some of the known facts were briefly outlined. Dynamical questions of an experimental nature are offered by the notion of center of resistance, and a problem in calculus of variations is that of determining the ogive of least resistance. That the distribution of pressure on the projectile in flight may be a complicated one was suggested by pointing out as one factor the rôle of the velocity of sound as affecting the dissipation of energy. A few descriptive remarks on the terminology and nomenclature of a projectile and a trajectory, with data on the German long range gun, concluded the paper. The rôle of the theory of probability, statistical methods, mechanical quadrature and graphical methods of solution were merely hinted at in passing.

(2) The pre-war maps of France are based on the Bonne conical projection with the meridian of Paris as primary meridian. Since this is an "equivalent" but not a "conform" representation, the distortions of angles and of distances at the Eastern frontier amount to 18′ and 1/379 respectively. Such errors are far from negligible for present-day gunnery.

The Lambert conical map representation is conform. This is an ideal pro-

jection for artillery fire, since the angles are preserved and the distortion in distances amounts at most to only 1/2037 for the maps of northern France. Moreover the Lambert representation permits of unlimited extension in longitude east or west of the primary meridian (meridian near Trier, Germany). On the map the "kilometer grid" with its Y-axis parallel to the primary meridian is printed. The x and y coördinates of any terrestrial mark are readily taken off the map. The "orientation" officer determines the x and y coördinates of the prospective position of the battery by measuring the horizontal angles between the lines from the battery B to three known terrestrial marks (three point problem). The "Lambert North" is obtained by a "round of the horizon." The Y-azimuth of the target T on which fire is to be opened and the distance BT are easily found from simple formulas of coördinate geometry.

(3) Miss Gould (daughter of the late Benjamin Apthorp Gould, the astronomer), who has been an instructor in the navigation courses at the University of Chicago, gave the results of a comprehensive study of the available books in navigation. Since the revival and extension of courses in navigation in American universities and colleges due to the war has made this a subject of interest to so many mathematicians, it is our hope that Miss Gould's discriminating criticisms may be made useful to a larger circle of readers through a fuller presentation of this paper in the columns of the MONTHLY.

(4) Professor Rietz called attention to the fact that there is simply an appalling amount of statistical work in the War Department, and the question that arises is not in regard to the magnitude of the statistical projects involved, but in regard to whether the statistical problems to be solved are of such a nature that a department of mathematics should give them special attention. During a reorganization of the Quartermaster Corps last February and March, the speaker had an opportunity to investigate the statistical methods in use in the War Department. This experience convinced him that a course might well be given in a department of mathematics that would have a useful place in the preparation of men for this branch of the service. At the University of Iowa during the past quarter, there has been given a very elementary course in statistics with special reference to the purpose just indicated. The course was taken by seventy-three men. The subject matter had to be adapted to the preparation of the average S. A. T. C. student, and to a time schedule of six clock hours per week. The time allotted to the course was divided between one class period and three hours per week in a statistics laboratory.

A notion can perhaps be obtained of the subject matter of the course as given the past quarter by the following general divisions of the ground covered: (1) tabulation of data, (2) frequency distributions and elements of probability, (3) graphical methods, and (4) averages. Along with the presentation in class of the meanings of simple, double, treble and quadruple tabulation, real applied problems for tabulation were given as laboratory exercises. The idea of a frequency distribution was next developed and the first laboratory exercise was to prepare a frequency distribution from the monthly rainfalls at Iowa City for

each month of the past twenty-five years. The student was required to write the answers to the following questions about resulting frequency distribution:

(1) Give reasons for the selection of your interval of sub-classes rather than some other interval.

(2) What is your estimate from this frequency distribution of the rainfall that is most frequent at Iowa City?

(3) What is your estimate of the relative frequency of a monthly rainfall in excess of three inches at Iowa City?

The meaning of probability was next taken up and a few of the elementary propositions leading up to the probabilities for repeated trials were developed. After this excursion of two or three days into probability, a return, by analogy, was made to frequency distribution of rainfall with a new light that at least suggests an explanation of the character of the frequency distribution of rainfall. Graphical methods of presentation of data were next discussed. The laboratory exercises required for the best presentations a rather large variety of curves and diagrammatic forms. In this connection, the graphing of a few simple mathematical functions was also given.

The extensive use of logarithmic paper in the statistical section of the Quartermaster Corps impressed upon the speaker the desirability of emphasizing the purposes for which this paper is adapted. The preparation of a sheet of logarithmic paper was given as a laboratory exercise. This was prepared without the use of a logarithmic table.

A start was made on the meanings and functions of different kinds of averages but there was no treatment of average in the laboratory work.

Professor Reilly, Mr. Taylor and the speaker did the teaching and prepared the material for this course. With regard to the results of the experiment, all feel that the laboratory has been very successful, considering the irregularities of attendance. The students were interested and did the work with enthusiasm. It is perhaps unnecessary to say that a course in the elements of statistics can be given much better when the students have at least freshman mathematics as preparation than under the conditions met in connection with this course. However, the work as a course in elementary statistics, with its war title deleted, is to continue for the remainder of this school year. The speaker then concluded as follows:

"To give some notion of the subject matter of the remainder of the course, let me say that it will include a treatment of different kinds of averages, and the functions used to describe dispersion in a frequency distribution. This will necessarily require a knowledge of logarithms and of interpolation at least by proportional parts. We shall certainly give more of probability theory than is contained in a good freshman text-book and draw illustrations from statistics, including the meaning of the normal probability curve. In the laboratory, we shall give exercises to test how nearly the normal curve fits some real frequency distributions of statistics. The course will include the preparation of index numbers and the rationale of different methods of averaging them. Finally, the course may include something of the meaning of correlation, and of the use and limitations of the correlation coefficient."

(5) Because of the limited time at his disposal, Major MacMillan stated only one of a number of important ordnance problems, this being the particular problem with which he has been occupied: A plane or dirigible is at an unknown height, is moving in an unknown direction with an unknown velocity, the force and direction of the wind and some other such data are incompletely known. The artillery officer has six seconds within which to make the necessary observations and compute the required direction and elevation of fire. He stated that with the best efforts of artillerists to improve the methods the guns hit these targets once in 15,000 shots. Then, commenting good-humoredly on the slender results which have followed from all the efforts of expert mathematicians and others, he added that a practiced gunner will without the use of these methods hit such targets once in about three or four thousand shots.

(6) The development of the methods of exterior ballistics in practical use was briefly sketched by Lieutenant Alger. In this development a series of schemes for taking account of air resistance have been devised, tested, and abandoned in turn. In general, a certain law of air resistance has been assumed, and tables and formulæ based thereon have been developed. In these formulæ a single constant, characteristic of the projectile, has been left to be determined by experiment. As long as this "constant" is invariant with changes in elevation or velocity within the limits of accuracy desired, a given method is useful. But each method in turn has come to a stage where the variations of the "constant" have been important. After vain attempts to derive laws for such variations, a new scheme has been adopted.

It was first assumed (by Newton) that the resistance of the air was proportional to the square of a projectile's velocity; on this basis the tables and methods of Otto and Euler were prepared. Better experimental knowledge led to the adoption of the cube law of Bashforth and later of the fourth power law of Zaboudski and Lardillon. Increased muzzle velocities necessitated more general laws, however, and after extensive experiments by Krupp, Mayevski and Hojel introduced a discontinuous law of resistance whereby different power laws are assumed to hold over several velocity ranges. In connection with these laws of resistance, various methods of solution of the differential equations of motion have been employed. The fact that, if the resistance is proportional to any power n of the velocity, all trajectories having the same initial inclination and initial retardation are similar, was used by Otto. Didion integrated the equation of the hodograph by taking a mean value α for the secant of the inclination assuming it constant throughout the integrations. Siacci made a better approximation by taking $\cos^{n-2}\theta/\cos^{n-1}\phi$ as constant and equal to a mean value throughout. This latter scheme in conjunction with the use of a pseudo velocity introduced by Siacci made a very satisfactory method for use in conjunction with Mayevski's laws. The method was reduced to a convenient basis by Braccialini and Ingalls, these having devised and tabulated secondary functions which could be computed by formal integration. These tables are still in use for rapid computation, but the modern use of high angle fire has rendered necessary new methods

for exact computation. This need has been met by the assumption of a smooth empirical law for air resistance and by the introduction of methods of solution of the equations by numerical integration. Such methods have long been in use but others have recently been developed in England by Littlewood and in America by Major F. R. Moulton. Improvements and extensions of the American method by G. A. Bliss, Capt. A. A. Bennett, and T. H. Gronwall have much reduced the labor of computation required. It is hoped the introduction of further improvements and the compilation of tables will enable the scheme to compare favorably with Ingalls's methods in rapidity as well as to exceed them in accuracy.

The methods of taking into account the variation of air density with altitude, the effects of wind, and the precession of a projectile, were not touched upon on account of lack of time.

(7) In order that a clear description of the problem might be possible, the following principle of relative motion was first stated by Professor Roever: If the motion of a particle with respect to axes fixed in space be known, the motion of the same particle with respect to a set of rotating axes can be determined. To the forces which account for the motion in the fixed system there must be added two forces in order to account for the motion in the rotating system. One of these forces acts along the perpendicular from the particle to the axis of rotation, its direction is away from the axis of rotation and its magnitude is $\omega^2 r$, where r is the distance of the particle from the axis and ω is the angular velocity of rotation of the second system with respect to the first. The second additional force is perpendicular to both the direction of motion of the particle (in the rotating system) and the axis of rotation; its magnitude is $2\omega v \cos \gamma$, where v is the velocity of the projectile along its path and γ is the angle which the direction of motion makes with a plane perpendicular to the axis of rotation. A rule for determining the sense of this force will be stated later.

The particle was then supposed to be the projectile of a gun. In the system which does not rotate with respect to the fixed stars, the only force which acts is that of the gravitational attraction of the earth for the particle, provided air resistance and wind effects are neglected. In the system which is at rest with respect to the rotating earth, the forces which must be brought into play in order to account for the observed motion are, in addition to the gravitational attraction just mentioned, the two forces referred to in the preceding paragraph. The resultant of gravitational attraction and the first additional force is a force whose magnitude is the weight. It is this resultant which determines the form of the level surfaces. To obtain the form of the path of the projectile (in the moving system) the second additional force must also be taken into consideration. When this force is not negelected we shall say that *rotation* has been taken into consideration. However, this force is usually assumed to be negligible. It is also usually assumed that the force, just defined, whose magnitude is weight, is constant in direction and magnitude in the region traversed by the projectile. We will call this the assumption *neglecting curvature*.

It was then shown how the trajectory (path of the projectile) which corre-

sponds to the assumptions neglecting rotation and curvature differs from that which takes into consideration rotation and curvature, for the case of both a vacuum and an atmosphere, the layers of constant density of which are level surfaces. Qualitatively the results may be stated as follows: The effect of rotation is the same as that of a wind which blows parallel to the plane of the equator and to the right (or left) of a person who is imagined to be walking on the north (or south) side of this plane along the path and in the direction of motion of the projection of the projectile on this plane. This effect is somewhat modified by air resistance. Since a local horizontal plane is, in general, not parallel to the plane of the equator, the effect of rotation, besides causing a drift of the trajectory, causes also a change in range in that it raises or lowers the trajectory. The effect of curvature is very slight. The differential equations of motion as well as those of the differential corrections due to rotation have been obtained independently by several investigators for the case where curvature is neglected. From the latter equations the drift and change in range were computed by a short arc process. From more general equations of motion obtained by the author, the differential corrections due to curvature were computed.

(8) The velocities encountered in trench mortar fire are not greater than 800 feet per second. The air resistance may accordingly be assumed proportional to the square of the velocity; and if we assume the flight of the projectile to take place in an atmosphere of uniform density, the various elements of the trajectory can be expressed in terms of quadratures, in which the integrands involve constants and the function $J = \int d\theta/\cos^3\theta$ algebraically if $\tan\theta$ be taken as the variable of integration. In order to integrate in simple terms we are obliged to find a convenient function of $\tan\theta$ to represent J with sufficient accuracy. For instance, Siacci's method is equivalent to writing $a\tan\theta$ for J in this case. Greater accuracy will be obtained by using the expression $a\tan\theta/(1 - b\tan\theta)$.

One of the important problems of ballistics is to determine the deviations of the trajectory due to a wind of uniform velocity and direction ("ballistic wind"). Now, it appears that the ratio of the change in distance along the range, to the deflection from the plane of fire (measured as a distance perpendicular to this plane), divided by the ratio of the corresponding wind-components, is very nearly a function of the angle of elevation φ of the gun only. The limiting value of this function is 7/3 and 1 for $\varphi = 0°$ and $\varphi = 90°$, and is in the neighborhood of 5/3 for $\varphi = 45°$.

Meeting of the Council of the Association.

The following twenty-one persons and one institution, on applications duly certified, were elected to membership:

To individual membership:

J. J. Arnaud, B.S. (Coll. of the City of New York). Master computer, Ord. Dept., Washington, D. C.

G. C. Autenrieth, A.M. (Columbia). Asst. prof., descr. geom. and drawing, Coll. of the City of New York.

MRS. KATHARINE D. BROWN, A.M. (Bucknell). Prof. of math., Drexel Inst., Philadelphia, Pa.

PIERCE BUTLER, Ph.D. (Hartford Theol. Sem.). Senior asst., The Newberry Library, Chicago, Ill.

MINNIE W. CALDWELL, A.M. (Missouri). Teacher of math., Marvin Coll., Fredericktown, Mo.

C. C. CARTER, Chapin, Ill.

H. S. EVERETT, A.M., Sc.M. (Bucknell). Asst. prof. of math., Bucknell Univ., Lewisburg, Pa.

W. J. FERGUSON, A.B. (Williams). With Ginn and Co., Chicago, Ill.

HAIG GALAJIKIAN, Ph.D. (Princeton). Range Firing Section, Aberdeen Proving Ground, Md.

H. M. GEHMAN, recently Aberdeen Proving Ground. Norristown, Pa.

ALICE BACHE GOULD, A.B. (Bryn Mawr). Instructor in navigation, Univ. of Chicago, Chicago, Ill.

A. E. LAMPEN, A.M. (Michigan). Prof. of math., Hope Coll., Holland, Mich.

E. P. LANE, Ph.D. (Chicago). Instr. in math., Rice Inst., Houston, Tex.

M. J. MCCUE, C.E., M.S. (Notre Dame). Prof. of civil engineering, Notre Dame Univ., Notre Dame, Ind.

JUSTIN NICOLET. Stud., extension dept., Univ. of Wisconsin. Chicago, Ill.

W. P. PARKER, A.M. (Davidson Coll.). Prof. of math., Union Chr. Coll., Pyengyang, Korea.

L. G. POOLER. Stud., Columbia Univ. New York, N. Y.

HARRIS RICE, B.S. (Worcester Polytech. Inst.). Instr. in math., Tufts Coll., Mass.

C. H. RICHARDSON, M.S. (Illinois). Prof. of math., Georgetown Coll., Georgetown, Ky.

R. B. STONE, A.M. (Harvard). Asst. prof. of math., Purdue Univ., West Lafayette, Ind.

C. W. WATKEYS, A.M. (Harvard). Prof. of math., Univ. of Rochester, Rochester, N. Y.

To institutional memership:

UNIVERSITY OF UTAH, Salt Lake City, Utah, to date from January, 1918.

On the unanimous recommendation of the Committee on Editor-in-Chief, the Council appointed PROFESSOR R. C. ARCHIBALD to this office, expressing regret that Professor Carmichael found it necessary to lay down the work which he has so ably carried during the past year and giving voice to the great satisfaction felt in being able to name Professor Archibald as his successor. Professor Archibald assumed the duties of his office with the January, 1919 issue of the MONTHLY.

As stated earlier in this report, the plan proposed by Professor Slaught for a committee to re-define the "units" in secondary mathematics was approved, and the question was referred to the standing Committee on Mathematical Requirements.

The Council transacted further business with respect to the *Annals of Mathematics* and the question of providing for certain expenses of the Committee on Mathematical Requirements which will be necessarily incurred in carrying out the important and far-reaching plans of that committee.

It was voted that the Council favor holding a summer meeting in conjunction with the American Mathematical Society.

The Council made the following appointments on the staff of the MONTHLY:

Committee on Publications:

R. C. ARCHIBALD, Editor-in-Chief.
W. A. HURWITZ.
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OTTO DUNKEL,	H. P. MANNING,	D. E. SMITH.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The secretary-treasurer announced the names of those just elected to membership by the Council. He also reported the death of the following eight members during the past year, all but Dr. McAtee having been charter members of the Association, and almost all having been long-time subscribers to the MONTHLY:
A. T. G. APPLE, Director Scholl Observatory, Franklin and Marshall College.
E. W. DAVIS, Professor of mathematics, University of Nebraska.
C. E. FLANAGAN, Conservative Life Insurance Company, Wheeling, W. Va.
R. A. HARRIS, U. S. Coast and Geodetic Survey.
CHRISTIAN HORNUNG, Professor of mathematics, Heidelberg University.
H. G. KEPPEL, Professor of mathematics, University of Florida.
ARTEMAS MARTIN, U. S. Coast and Geodetic Survey.
J. E. MCATEE, Instructor in mathematics, University of Illinois.

In connection with the list of mathematicians in war service published by the secretary-treasurer in the MONTHLY for January, 1919, it was announced that the names of 110 members of the Association were known to the secretary-treasurer as having been enrolled in national service, including Y. M. C. A. and other non-combatant branches; this is a record of which the association may justly be proud. An instance worthy of mention is that of one of the members of the Association, J. W. DAPPERT of Taylorville, Illinois, in whose family are listed four lieutenants (one deceased), a sergeant in Europe, and a daughter in the War Risk Bureau.

The election of officers for the year 1919 was conducted by mail and in person at this meeting, as provided by the constitution. The tellers (A. B. Coble and E. B. Stouffer) appointed by the Council reported the result of the balloting as follows, 284 ballots having been cast, some of which were blank in part:

For President: H. E. Slaught, 219 votes; J. W. Young, 65 votes.

For Vice-President: H. L. Rietz, 153 votes; R. G. D. Richardson, 146 votes; D. E. Smith, 137 votes; Alexander Ziwet, 122 votes.

For additional members of the Council to serve until January, 1922: E. V. Huntington, 186 votes; E. H. Moore, 183 votes; L. P. Eisenhart, 148 votes; B. F. Finkel, 140 votes; H. F. Blichfeldt, 139 votes; W. H. Roever, 118 votes; D. A. Rothrock, 110 votes; J. N. Van der Vries, 99 votes.

The following were accordingly declared elected:

President, H. E. SLAUGHT, University of Chicago.

Vice-Presidents, R. G. D. RICHARDSON, Brown University, and H. L. RIETZ, University of Iowa.

Additional members of the Council to serve until January, 1922:

L. P. EISENHART, Princeton University,
B. F. FINKEL, Drury College,
E. V. HUNTINGTON, Harvard University,
E. H. MOORE, University of Chicago.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to December 2, 1918. The report of the auditing committee (Mary E. Sinclair, H. E. Slaught, and C. N. Moore, chairman) was then made, and both reports were accepted and approved. The financial report is printed in full below.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 2, 1918.

RECEIPTS.

Balance Dec. 1, 1917		$3,485.47
1917 subscriptions	$ 5.70	
1917 indiv. dues	94.00	
1917 instit. dues	13.00	
1918 subscriptions	438.70	
1918 indiv. dues	2,803.38	
1918 instit. dues	267.25	
Initiation fees	142.00	
Sale copies of MONTHLY	14.09	
Sale reprints	13.96	
Advertising	631.25	
Exchange	.94	
Interest State Savgs. Bk	85.29	
Interest Peoples Bk	36.65	
Interest Liberty Bond	20.00	
Total 1918 receipts		4,566.21
Total receipts up to 1919 business		$8,051.68
Total expenditures		4,539.84
Balance on 1918 business		$3,511.84

EXPENDITURES.

Publisher's bills		$2,953.76
Membership list		232.76
Managing editor's office		79.76
Editor-in-chief's office		77.08
Other editors' postage		29.12
Committee on Membership		45.89
Com. on Math. Requirements		112.13
Com. on Libraries		2.61
Dept. Undergraduate Math. Clubs		16.89
Secretary-Treasurer's office:		
Postage	$161.17	
Bond	5.00	
Office supplies	14.15	
Typewriter	45.00	
Express, telegrams, freight, etc.	16.78	
Library expense	1.00	
Clerical work	217.62	
Printing	158.54	
Chicago meeting	36.59	
Dartmouth meeting	80.80	
Paid to sections from initiation fees	24.51	
Safety deposit rental	1.67	
		762.83
Annals subvention		225.00
Interest on Liberty Bond		2.01
Total expenditures		$4,539.84

Received on 1919 business	216.27	Cash on hand, not deposited	$ 37.85
		Checking account	545.44
		State Savgs. Bk. Co. account	1,564.77
		Peoples Bkg. Co. account	1,080.05
		Liberty Bond	500.00
Balance Dec. 2, 1918	$3,728.11	Bank balance Dec. 2, 1918	$3,728.11

Approved by the auditing committee,

C. N. MOORE, *Chairman*,
MARY EMILY SINCLAIR,
H. E. SLAUGHT for the Council,
Committee on Finance.

December 28, 1918.

When the accounts were closed on December 2, 1918, as was necessary in order to furnish the auditing committee a complete record, there remained on the total business for the year 1918 the following items:

BILLS RECEIVABLE.	
Advertising	$150.00
1918 dues unpaid	75.00
	$225.00

BILLS PAYABLE (either paid in December or estimated).	
Publisher's bills, Sept.–Dec.	$1,450.00
Sept. and Dec. *Annals* subvention	150.00
Init. fees due to sections	60.00
President's office	10.00
Manager's office	20.00
Editor-in-Chief's office	30.00
Other editors' postage	25.00
Secretary-treasurer's office	150.00
Printing annual ballot, program, etc.	75.00
Additional postage	50.00
	$2,020.00

If to the balance on 1918 business shown in this report, $3,511.84, there be added the amount of bills receivable, $225.00, and there be subtracted the estimated amount of bills payable, $2,020.00, there results an estimated final balance on 1918 business of approximately $1,700.00. It will again be recalled that about $1,000 of this surplus was the amount turned over to the Association by the management of the MONTHLY when the Association was formed, a fund which the Council feels must be held intact as a reserve fund. The financial gain made the past year is very gratifying, in view of the fact that we must face the coming year a certain decrease in the returns from advertising, and an almost inevitable increase in the item of printing if the MONTHLY is to remain at its present status, or, as we hope, to develop still further in its inspiring task of stimulating and strengthening American mathematics.

W. D. CAIRNS, *Secretary-Treasurer.*

ON THE ENVELOPE OF THE WALLACE LINES OF AN INSCRIBED QUADRANGLE.

By DAVID F. BARROW, University of Georgia.

It has long been known that the Wallace lines of a triangle envelop a three cusped hypocycloid.[1] Steggall[2] has investigated the envelope of the Wallace lines of an inscribed polygon. He gives the equation of the Wallace line in the general case and some facts about regular polygons. All he does with the quadrangle is to give the condition that the envelope be a four-cusped hypocycloid. We shall examine this case in more detail.

If perpendiculars be dropped upon the sides of a triangle from a point on the circum-circle, the three feet are collinear on a line called the Wallace line of the point with regard to the triangle. If we take an inscribed quadrangle, and omit each vertex in turn, we obtain four triangles; and the feet of the perpendiculars dropped from any point of the circle upon its four Wallace lines with regard to these triangles lie on a line called the Wallace line of the point with regard to the quadrangle. The process is capable of indefinite extension.

Let A_1, A_2, A_3, A_4 be the vertices of a quadrangle inscribed in a circle of radius R, and P any point on the circle. With the origin at the center, let θ_i and θ denote the angles measured from the positive x-axis around the circumference to A_i and P respectively. Then the equation of the Wallace line of P with regard to the quadrangle is[3]

$$\begin{aligned} x\cos\tfrac{1}{2}[\theta_1+\theta_2+\theta_3+\theta_4-2(\theta-\pi)] + y\sin\tfrac{1}{2}[\theta_1+\theta_2+\theta_3+\theta_4-2(\theta-\pi)] \\ = R\{-\cos\tfrac{1}{2}[\theta_1+\theta_2+\theta_3+\theta_4-4(\theta-\pi)] \\ + 2\sin\tfrac{1}{2}(\theta-\theta_1)\sin\tfrac{1}{2}(\theta-\theta_2)\sin\tfrac{1}{2}(\theta-\theta_3)\sin\tfrac{1}{2}(\theta-\theta_4)\}. \end{aligned} \tag{1}$$

Let the origin be moved to the point whose coördinates are

$$\tfrac{1}{4}R(\cos\theta_1+\cos\theta_2+\cos\theta_3+\cos\theta_4),\ \tfrac{1}{4}R(\sin\theta_1+\sin\theta_2+\sin\theta_3+\sin\theta_4)].$$

This is the point of concurrence of the three lines joining mid-points of pairs of opposite connectors of the quadrangle. Furthermore let $\theta_1+\theta_2+\theta_3+\theta_4=\pi$, which may be done without loss of generality by a proper choice of the direction of the x-axis. All this causes the equation of the Wallace line to reduce to

$$\begin{aligned} x\sin\theta + y\cos\theta = \tfrac{1}{4}R[3\sin 2\theta - \cos\tfrac{1}{2}(\theta_1+\theta_2-\theta_3-\theta_4) \\ - \cos\tfrac{1}{2}(\theta_1-\theta_2+\theta_3-\theta_4) - \cos\tfrac{1}{2}(\theta_1-\theta_2-\theta_3+\theta_4)]. \end{aligned} \tag{2}$$

[1] *Steiner*, "Gesammelte Werke," pp. 641–647, "Über eine besondere Curve dritter Classe (und vierten Grades)." Other articles have been written about it, but this is probably the first.

[2] J. E. A. Steggall, "On the envelope of the Simson line of a polygon." *Edinb. Math. Soc. Proceedings*, Vol. 14, p. 122–126 (1896).

[3] This equation is taken from Steggall, *loc. cit.*, but with a change of notation.

Now θ is the parameter in this equation, and on differentiating with regard to θ we obtain

$$x \cos\theta - y \sin\theta = \tfrac{3}{2}R \cos 2\theta. \tag{3}$$

Now (3) is a line which meets (2) at the point where (2) touches its envelope. Moreover (3) is evidently orthogonal to (2), hence (3) envelops the evolute of the envelope of (2). But we recognize[1] that (3) envelops a hypocycloid of four cusps.

THEOREM 1. *The envelope of the Wallace lines of an inscribed quadrangle is an involute of a four cusped hypocycloid whose cusps lie on a circle of radius three times that of the circum-circle.*

Next let an angle α be defined by the equation

$$-3 \sin 2\alpha = \cos \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 + \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 - \theta_3 + \theta_4).$$

Equation (2) may then be written

$$x \sin\theta + y \cos\theta = \tfrac{3}{2}R \sin(\theta + \alpha) \cos(\theta - \alpha).$$

Transform to oblique axes by

$$x = x' \cos\alpha + y' \sin\alpha,$$

$$y = x' \sin\alpha + y' \cos\alpha,$$

which turns the x-axis through an angle α and the y-axis through an angle $-\alpha$. This gives

$$\frac{x'}{\tfrac{3}{2}R \cos(\theta - \alpha)} + \frac{y'}{\tfrac{3}{2}R \sin(\theta + \alpha)} = 1$$

The intercepts on the axes are read off from the denominators in this equation, and since the angle between the axes is $\pi/2 - 2\alpha$, we easily calculate the length intercepted on the line by the axes to be $\frac{3}{2} \cdot R \cos 2\alpha$.

THEOREM 2. *There are two Wallace lines of an inscribed quadrangle which intercept a constant segment on all the others. These two pass through the point of concurrence of the three lines joining the mid-points of pairs of opposite sides of the quadrangle.*

This gives a very neat generation of the Wallace lines by allowing a constant segment to move with its ends sliding along two oblique lines. If, however, $\alpha = \pm \pi/4$ this generation becomes illusory since the two oblique lines would coincide. But from it we can get another generation which is never illusory, as follows: If a circle rolls on the interior of another circle of twice its radius, any point on the circumference of the rolling circle traces a diameter of the fixed circle. Therefore a chord of the rolling circle will be a line of constant length

[1] See Williamson's Differential Calculus, seventh edition, New York (1889), page 347, formula (18).

moving with its end points on two oblique lines. If the chord becomes a diameter the oblique lines become perpendicular, and if the chord becomes tangent the oblique lines coincide.

THEOREM 3. *The Wallace lines of an inscribed quadrangle are the successive positions of a chord of a circle which rolls upon the interior of a circle of twice its radius.*

In trying to picture the curve imagine first a hypocycloid of four cusps, which is one extreme case corresponding to $\alpha = 0$. Then suppose two opposite arches to grow longer at the expense of the other two which are shortened, so that the cusps stand at the vertices of a rectangle. This will be an intermediate figure. To get the other extreme let the two opposite arches swell till they become tangent to each other at the center of the curve. This corresponds to $\alpha = \pm \pi/4$. If all the vertices of the quadrangle approach coincidence the figure approaches this shape, but this is not the only case giving such a shape. It should be noted that not every involute of a four-cusped hypocycloid can be the envelope of the Wallace lines of a quadrangle if we confine ourselves to real quadrangles.

We have seen that when $\alpha = 0$ the envelope is a four-cusped hypocycloid. In terms of the θ_i this condition is

$$\cos \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 + \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 - \theta_3 + \theta_4) = 0.$$

A rotation of the axes has the effect of adding a constant to each θ_i, which obviously leaves this condition unchanged. We may therefore suppose that the axes have such a direction that $\theta_4 = 0$. Then the condition may be manipulated into the form

$$3 \cot \tfrac{1}{2}\theta_1 \cot \tfrac{1}{2}\theta_2 \cot \tfrac{1}{2}\theta_3 + \cot \tfrac{1}{2}\theta_1 + \cot \tfrac{1}{2}\theta_2 + \cot \tfrac{1}{2}\theta_3 = 0. \tag{4}$$

This looks familiar and reminds us of the following fact. If B_1, B_2, B_3 are the vertices of an equilateral triangle and Q any other point in the plane, and if φ_i denotes the angle made by the line QB_i with the bisector of the interior angle of the triangle at B_i, then

$$3 \tan \varphi_1 \tan \varphi_2 \tan \varphi_3 + \tan \varphi_1 + \tan \varphi_2 + \tan \varphi_3 = 0.$$

This shows that if we take $\theta_i = \pi - 2\varphi_i$ $(i = 1, 2, 3)$, condition (4) will be satisfied, which suggests how to solve the following

Problem. Given three points on a circle, to construct a fourth point thereon so that the Wallace lines of the quadrangle formed by the four points shall envelop a four cusped hypocycloid.

Call the given points A_2, A_3, and A_4; and measure the angles from A_4 to A_3 and from A_4 to A_2 around the circumference of the circle. Lay off half the supplements of these angles from the bisectors of two of the angles of an equilateral triangle. Join the third vertex to the intersection of the two lines thus constructed. Take the angle formed by this last line and the bisector of the interior

angle at the third vertex, and lay off twice its complement from A_4 around the circumference of the circle. This locates the desired point A_1 to complete the quadrangle.

Two special cases of quadrangles which yield four cusped hypocycloids deserve mention. We easily prove by condition (4)

THEOREM 4. *If an inscribed quadrangle is a square, or if three of its vertices form an equilateral triangle, then its Wallace lines envelop a four-cusped hypocycloid.*

Suppose we cause the point P to remian fixed while the quadrangle $A_1A_2A_3A_4$ revolves about the center of the circum-circle without changing its shape. This can be done if in equation (1) we let θ be constant and replace each θ_i by $\theta_i + \lambda$. Thus the θ_i will be initial values and λ the angle through which the quadrangle has rotated, λ being now the parameter. Then by methods similar to those we have used it is not difficult to prove

THEOREM 5. *If an inscribed quadrangle revolves about the center of its circumcircle, the Wallace line of a fixed point on the circumcircle envelops a curve whose evolute is in general a two-cusped epicycloid; but if the quadrangle is a rectangle the envelope is a circle, and if it is a square*[1] *the envelope is a point. The successive positions of the Wallace line can be described as the successive positions of a line rigidly attached to a circle which rolls upon the exterior of a fixed circle of equal radius.*

It seems a pity not to generalize some of these theorems to the case of an inscribed n-gon, but only theorem 4 seems easy to extend. Steggall has found the envelope in the case of a regular n-gon to be an n-cusped hypocycloid, which generalizes part of theorem 4; and the present writer has generalized the remainder of this theorem so that it reads:

THEOREM 6. *If an inscribed n-gon is regular, or if $n - 1$ of its vertices form a regular polygon, its Wallace lines envelop an n-cusped hypocycloid.*

GEOMETRIC EXPLANATION OF A CERTAIN OPTICAL PHENOMENON.[2]

By WM. H. ROEVER.

Description of the Phenomenon.—In the parcel checking-room of the new Union Station at Kansas City, Missouri, there is a counter covered with brass plates which have, during the course of time, received numerous scratches by the baggage which is moved around upon the counter. The scratches are not very deep and they seem to be of fairly uniform distribution in both density and direction, as one might expect them to be after the cause of their formation has been in operation for some time. The baggage room is lighted by large electric lamps which are not very close together, so that an observer near the counter may regard the illumination in his immediate neighborhood as being due to a

[1] Steggall proves that for any regular n-gon the envelope is a point, *loc. cit.*

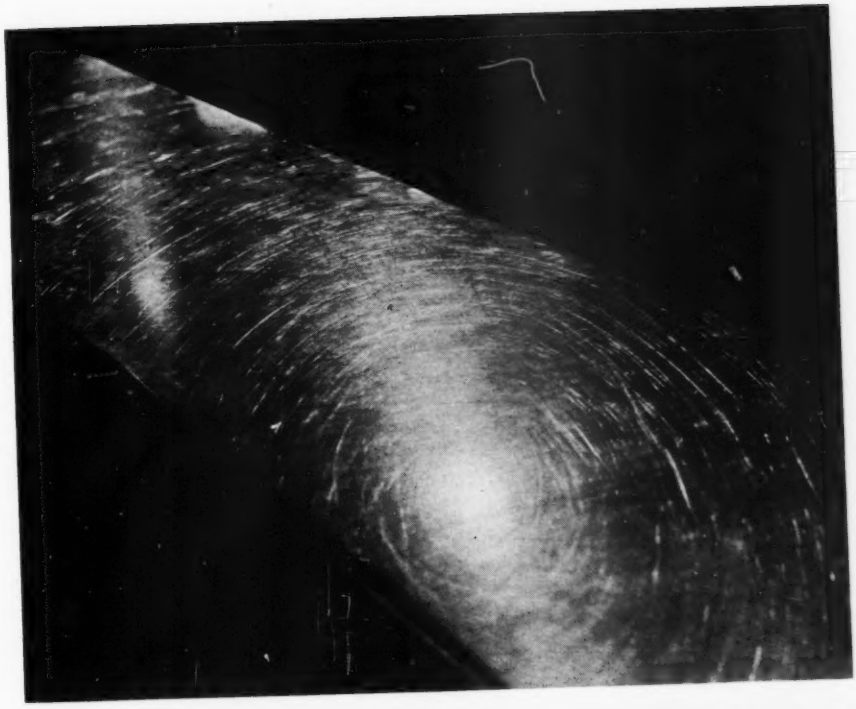
[2] Presented to the American Mathematical Society (Southwestern Section), December 1, 1917.

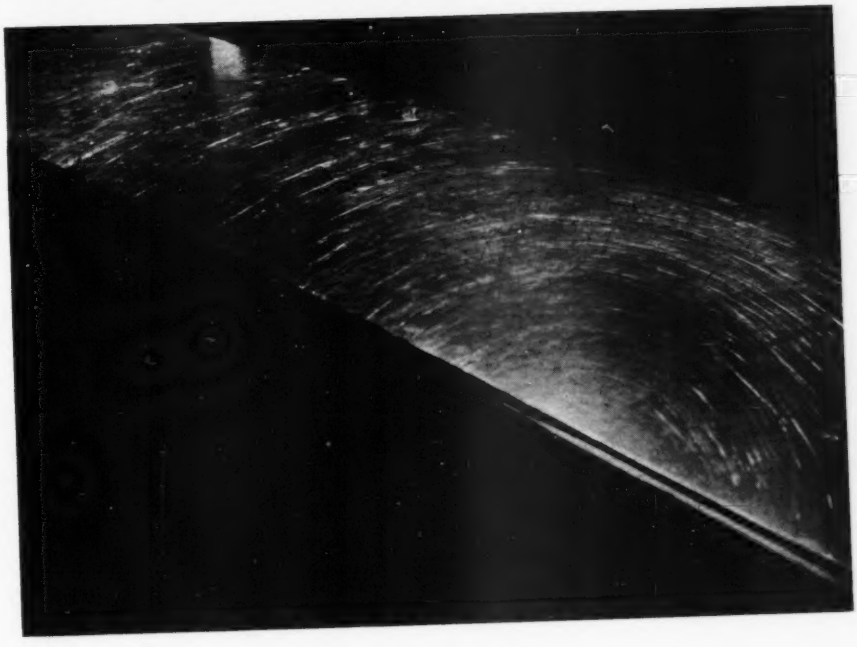
single lamp. Notwithstanding the apparently lawless nature of the manner of formation of these scratches, an observer anywhere near the counter and regardless of the direction of the illuminating electric light, will observe what appears to be a one-parameter family of illuminated ellipses which are approximately concentric and similar (see the accompanying pictures on the opposite page).

The Explanation.—A prolate spheroid, *i. e.*, an ellipsoid obtained by revolving an ellipse around its major axis, has the property that each of its points is a brilliant point with respect to its foci. In other words the focal radii to any point of the surface make equal angles with the normal to the surface at that point, and the normal lies between and in the plane of the focal radii. Consequently any reflecting surface or curve[1] which is tangent to such an ellipsoid of which a point source of light and an observer's eye are the foci, will appear to have at its point of contact with the ellipsoid a luminous point, *i. e.*, a brilliant point. The electric light and the eye of the observer are the foci of a one-parameter family of confocal ellipsoids of revolution. These ellipsoids intersect the plane of the brass-covered counter in a one-parameter family of ellipses (which are neither concentric nor similar in general, but are approximately so for the smaller curves of the family). For different positions of the observer's eye (and of the lamp, which, however, is fixed) there are, of course, different families of ellipses on the counter. Those scratches on the counter which are tangent to the members of this one-parameter family of ellipses, will have brilliant (or luminous) points at their points of contact with these curves. Owing to the fact that a scratch may have some curvature and some width and also because the source of light is not a point, it follows that not merely a point but that a small arc of the scratch becomes illuminated. These small illuminated portions of scratches (though short and disconnected) are well distributed, and even though few may lie along any individual ellipse, they do, in the aggregate, give the general impression of a one-parameter family of illuminated ellipses, *i. e.*, they make visible, so to speak, the geometric ellipses described above in much the same way that iron filings distributed in a magnetic field make visible the lines of force of that field.[2]

[1] We will regard as a curve the exterior surface of a wire of small cross section, or the gutter-like surface of a scratch.

[2] This brilliant point phenomenon is different from any of those described by the author in the *Transactions of the American Mathematical Society*, Vol. 9, No. 2, pp. 245–279; *Bulletin of the American Mathematical Society*, Vol. XXII, No. 5, p. 218; THE AMERICAN MATHEMATICAL MONTHLY, Vol. XX, No. 10, pp. 299–303, and Vol. XXI, No. 3, pp. 69–77.





"CONCERNING A METHOD FOR FINDING A PARTICULAR INTEGRAL." NOTE ON PROFESSOR COBLE'S ARTICLE.[1]

By HENRY P. MANNING, Brown University.

The method of undetermined coefficients for obtaining the particular integral in certain cases of linear differential equations is much easier and shorter than the other methods given in our text-books, except perhaps some of the mysterious symbolic methods, and it is at least as short as these. Professor Coble expresses the opinion[2] that Cohen was the first to publish it, and this may be true so far as English and American texts are concerned. Neither Johnson, nor Murray, nor Forsyth mention it. The method, however, is to be found in the second edition of the third volume of Jordan's *Cours d'Analyse*[3] which appeared some years earlier. Cohen explains the theory a little differently, but Jordan's explanation, though brief, is very clear and much simpler than Cohen's. It is so simple, in fact, that with a little elaboration it can be taught to undergraduate academic and engineering students, even when not in their books, and it can be applied not only to forms of second member that occur most frequently when the equations have constant coefficients, but also to cases of the so-called homogeneous or Cauchy equations, and to some other types that occur. It is analogous to the method of differentiation employed for certain equations that are not linear. From a given linear differential equation is obtained an equation of higher order with the second member zero, and, when the complementary function of this equation can be obtained, that part which does not belong to the complementary function of the given equation is determined by substitution so as to satisfy it as a particular integral. In practice we usually know the form of the result and have only to substitute and determine the constant coefficients.[4]

RECENT PUBLICATIONS.

REVIEW.

Analytic Geometry. By EDWIN S. CRAWLEY and HENRY B. EVANS. Philadelphia, E. S. Crawley, 1918. 12mo. 14 + 239 pages. Price, $1.60.

This book begins with seven preliminary pages given to review formulas from Algebra and Trigonometry and a set of tables, one of the three place natural values of the trigonometric functions of angles measured in radians and degrees for each degree from 0^0 to 90^0; another of three place common logarithms of numbers from 10 to 99; another of two place Napierian logarithms of numbers from 0 to 10.9; and two tables for Napierian anti-logarithms, one for positive and the

[1] In this MONTHLY, January, 1919, pages 12–15.

[2] In a footnote on page 13.

[3] C. Jordan, *Cours d'Analyse de l'Ecole Polytechnique*, Tome 3, 2d ed., Paris, 1896; arts. 132–133, pp. 160–163.

[4] C. E. Love explains this method briefly (*Calculus*, New York, 1916, pp. 313–315).

other for negative powers of e; and also a table of square roots of numbers from 0 to 99 to two decimal places. These tables precede a text of twelve chapters (235 pages) of which the first eleven are devoted to plane analytic geometry and the twelfth to an "Extension of coördinate geometry to some space problems." The text is followed by an appendix (four pages) wherein are sketched twenty of the ordinary curves "for reference," used in analytic geometry and calculus.

The subject matter treated is that of the text books now in common use with the addition of some elementary discussion of imaginary elements in geometry. The imaginary unit is introduced on page *viii* of the Introduction: "$\sqrt{-1}$ is called the imaginary unit and is usually denoted by the letter i."

On pages 20–22 in connection with a discussion of the curve $4x^2 + 9y^2 = 36$, imaginary points on the curve are defined as points which cannot be constructed but which correspond to pairs of numbers (one of them being imaginary in the illustrative example) satisfying the equation. The text does not state whether or not both of the numbers may be imaginary.

On page 30 we find: "*Definition. An equation which is satisfied by only imaginary values of the variable, or by at most a finite number of real values, is called the equation of an* **IMAGINARY LOCUS.**" Three types of equations fall under this classification: "(*a*) Equations one or more of whose coefficients are imaginary numbers; (*b*) Equations all of whose coefficients are real, but which are not satisfied by any real values of the variables; and (*c*) Equations all of whose coefficients are real, but which can be satisfied by only a limited number of pairs of real values of x and y." On pages 58–60 in the chapter on the straight line, under the head of "Imaginary Points and Lines," are the definition: "*Two imaginary points whose coordinates differ only in the signs of the imaginary parts are called* **CONJUGATE IMAGINARY POINTS,**" and the theorems: "*If an imaginary point lies on a real line* $Ax + By + C = 0$, *then the conjugate imaginary point lies on the same line*"—"*On every imaginary line there is one and only one real point,*"—and "*A pair of conjugate imaginary lines meet in a real point.*" On page 67 the imaginary circle is defined as the locus of $x^2 + y^2 + Lx + My + N = 0$ when $N > 1/4(L^2 + M^2)$.

These definitions and theorems have a few exercises following them and depending upon them. But the authors seem to have made no further use of the imaginary number in its geometric interpretation. While there is an advantage in including this material for the purpose of keeping the meaning of the imaginary number before the student, it is very doubtful if this advantage is sufficient to make up for the difficulty thereby added to the course, particularly since the greatest use of the material cannot come into play until the student reaches the study of the complex variable. The discussion is so brief that there is no mention of even the circular points at infinity. With such a brief treatment of imaginaries the student is bound to wonder what they were put in the book for. Why introduce them if they are not to be used? The preface explains by saying that "owing to the increasing use of the imaginary and its growing importance to the student of pure and applied mathematics some elementary discussion of imagi-

nary elements in geometry has been included, which the authors believe will be of value in accustoming the student to look upon the complex number as a useful member of the number system." The reviewer doubts that the brief treatment in the book really teaches any of the *uses* of the imaginary number.

The arrangement and form of presentation of the subject matter are planned to lessen the difficulties which confront the student in his transition from elementary algebra and geometry, where memory has too often been sufficient for his needs, to a subject wherein the interpretation of equations and loci and the understanding of mathematical relations is necessary to any sort of mastery of the work. This transition marks the end of mathematical interest to a large number of students each year and the text before us as an attempt to lessen these transition difficulties seems entirely justified in spite of the fact that analytic geometries are already so numerous. The treatment generally followed in the text differs from the customary form of presentation chiefly in the putting in the form of a problem much of the material which is usually put in the form of a theorem. Thus in the chapter on the straight line the book first sets down on page 40 "Elementary geometry teaches that a straight line is determined by two conditions properly chosen." This is followed by the "*Problem. To derive the equation of a straight line in terms of the coördinates of two given points on the line.*" All the standard forms of the equation of the straight line are thus obtained as "problems." Later in the book we find the

"*Problem. To find the equation of the ellipse referred to rectangular axes with the principal axis for x axis and the origin at the center of the ellipse.*" The putting of such material in problem form makes the student feel that he is being let into the secret of the methods by which analytic geometry was built up. It is often discouraging to the student to be shown a finished demonstration without any clue to the original method of its derivation.

Especially worthy of note is Chapter XI (14 pages) in which the authors present "Empirical equations" in two cases, first, when the law of the curve is given, and second, when the law of the curve is not given. This topic is presented by means of illustrative examples based upon experimental data. The first problem is to determine the equation representing the relation of elongation to load in an actual piece of steel. Here it is assumed known that the elongation is proportional to the load and therefore that the relation is linear, so that the locus is $y = mx + b$ where y is the load and x the elongation. It remains to choose two points determined by experimental data through which a line shall pass containing as many points as possible and leaving the points not on the line equally distributed above and below it. The two chosen points serve to determine the m and b of the equation.

In the second problem, when the law is not given, the points determined from experimental data are first plotted and an equation whose locus closely approximates the true locus is chosen. The equation suggested for general use is

$$y = a + bx + cx^2 + dx^3 + \cdots$$

where a, b, c, etc., are determined from the equation by giving x and y the values obtained by experiment.

This process, now commonly found in text books on analytic geometry, has a definite application in experimental sciences and ought to be given if the time is available. It furnishes also a convenient setting for a discussion of continuity.

The last chapter of the book (31 pages) is a successful adaptation of the conceptions and equations of analytic geometry of three dimensions to the needs of the student in his first course in calculus without attempting to develop fully the subject of solid analytic geometry.

The authors have not seen fit to introduce any formal treatment of limits or continuity, although they define a tangent as the limiting position of a secant, and suggest the notion of continuity in the chapter on "Empirical equations." It may be a debatable question how much of this should be given in a first course. Some teachers would be satisfied with the loose notions of elementary geometry while others would demand the precise ϵ, δ definitions, but it does seem that theories so involved in analytic geometry should be formally mentioned, and an attempt made to give to the student conceptions as accurate as his ability to understand permits.

On pages 53–56 the authors discuss the "Perpendicular distance to a given point from a given line." The discussion is based upon the unnecessary Hesse normal form of the equation of the straight line. Professor Maxime Bôcher, in his *Plane Analytic Geometry* (New York, 1915), first obtains the formula on pages 38 and 39; and on page 43 he sets up the Hesse normal form in fine print. In this MONTHLY, December, 1917, page 476, Mr. R. M. Mathews finds the distance formula without making use of the Hesse normal form. Similarly, in this MONTHLY for April, 1918, page 181, Mr. H. T. Burgess comments favorably upon the derivation of Mr. Mathews, gives still another derivation, and states that "Hesse's normal form in this connection may well be relegated to Professor Miller's collection of 'Obsoletes.'"[1]

The book is well supplied with carefully selected exercises graded into "Normal exercises" and "General exercises." On the whole the book is a very teachable class room text which should enable the student to grasp the subject matter and use it successfully in the many well selected problems of the text and in other elementary problems of analytic geometry.

Strange to say, the book has no index. This seems inexcusable in a modern text.

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NOTES.

Simplified navigation for ships and circraft. A text book based upon the Saint Hilaire method. By C. L. POOR. New York, Century, 1918. 12mo. 18 + 126 pp. + 1 plate ["The line of position computer"]. Price $1.50.

[1] See also Maxime Bôcher, "Concerning Direction Cosines and Hesse's Normal Form." AMERICAN MATHEMATICAL MONTHLY, September, 1918, pages 308–10.

Contents—Chapter I: Finding one's position at sea and in the air, pp. 3–22; II: Determination of the "D. R." altitude and azimuth, 23–71; III: Corrections to be applied to the measured altitude, 72–88; IV: Simultaneous altitudes, 89–98; V: Notes and practical applications, with special reference to aërial navigation, 99–112; Tables: 113–126.

The *Journal of the Engineers' Club of Philadelphia* for June and July, 1918, contains an article by Carl G. Barth entitled "The income tax: an engineer's analysis with suggestions." The author finds a mathematical formula which agrees closely with the tax imposed by the present law and which would "smooth out" the inequalities in the gradation of this schedule. The formula is compared with Pareto's law of incomes in its estimate of the total national income of the United States; and certain criticisms of the details of the law are made, *e. g.*, that of exemption, on the mathematical basis afforded by the formula.

Four-place logarithmic and trigonometric tables together with interest tables—edited by L. C. KARPINSKI.[1] Ann Arbor, Mich., G. Wahr, 1918. 8vo. 30 pp. Stiff paper, price 30 cents.

Contents—Explanation of tables, pp. 5–7; Constants with their logarithms, 9; Logarithms of numbers, 100–999, 10–11; Logarithms of numbers, 1000–2009, 12–13; Logarithms of sines and cosines, 10′ intervals, 14–15; Logarithms of tangents and cotangents, 10′ intervals, 16–17; Logarithms of sines and tangents by minutes, 0° to 14° (Logarithms of cosines and cotangents by minutes 76° to 90°), 18–19; Numerical values of sines and cosines, 10′ intervals, 20–21; Numerical values of tangents and cotangents, 10′ intervals, 22–23; Squares, cubes, square-roots, cube-roots, and reciprocals, 1 to 100, 24; Degrees and minutes expressed in radians, Values of e^x and e^{-x}, 25; Accumulation of 1 at the end of n years, r^n, 26; The present value of 1 in n years, v^n, 27; The accumulation of an annuity of 1 per annum at the end of n years, 28; The present value of an annuity of 1 per annum for n years, 29; The annual sinking fund to accumulate to 1 at the end of n years, and by addition of i the annuity which 1 will purchase, 30.

The Metric System. A brief explanation, with Exercises and Tables of Equivalents; also a brief discussion of the Centigrade Thermometer and Foreign Money is the title of a pamphlet of sixteen pages (about 10 × 14 cm.), written by Professor J. W. YOUNG, and published last December for the Education Bureau of the National War Work Council of Young Men's Christian Associations by Association Press, 347 Madison Avenue, New York. Thirty-five thousand copies have been distributed already. [The scale of a decimeter on the edge of the outside cover is curiously inaccurate—about a millimeter too long.]

The same press is already at work upon a very short course in algebra and geometry, which is to give just enough of these subjects to enable a man to take up the study of trigonometry. This pamphlet was prepared by Professors YOUNG and MORGAN who are also at work on a very short trigonometry, to be published by the Association Press. It was originally intended that these publications should constitute a basis of instruction in Y. M. C. A. huts at army and navy camps all over the country. The need of something of the kind was everywhere keenly felt.

Amusements in Mathematics by H. E. DUDENEY. London, Edinburgh, and New York, T. Nelson and Sons, 1917. 8vo. 8 + 258 pp. Price 3s. 6d.

[1] These tables are very similar to those at the end of *Unified Mathematics* by L. C. Karpinski, H. Y. Benedict, and J. W. Calhoun. Boston, Heath, 1918.

Contents—Arithmetical and algebraical problems, pp. 1–27; Geometrical problems, 27–56; Points and lines problems, 56–58; Moving counter problems, 58–68; Unicursal and route problems, 68–76; Combination and group problems, 76–85; Chessboard problems, 85–109; Measuring, weighing, and packing puzzles, 109–111; "Crossing river" problems, 112–114; Problems concerning games, 114–117; "Puzzle] games," 117–119; Magic square problems, 119–127; Mazes and how to thread them, 127–137; The paradox party, 137–141; Unclassified problems, 142–148; Solutions, 148–252; Index, 253–258.

This exceedingly interesting volume, packed with descriptions in small type and illustrated by hundreds of wood-cuts, would be a marvel of cheapness even in peace times.

Cours de mécanique professé à l'Ecole Polytechnique. Par L. CORNU. Tome 3. Paris, Gauthier-Villars, 1918. Royal 8vo. 2 + 669 pp.

Contents—Livre X: Résistance des matériaux, pp. 1–170; XI: Hydraulique, 171–310; XII: Thermodynamique, 311–428; XIII: Théorie des machines, 429–634; XIV: Notions d'aviation, 635–654.

Quotation from the "Avertissement": "Ce troisième et dernier Volume traite de la Mécanique appliquée. Les circonstances que nous traversons m'ont déterminé à entrer dans des développements dépassant les limites du Cours actuel de l'Ecole Polytechnique: tout porte à croire, en effet, que, la paix revenue, une transformation profonde s'opérera dans les études scientifiques, qui devront s'adapter plus étroitement aux réalités de la vie, en vue de mieux armer les Français pour la lutte économique succédant aux combats meurtriers . . . Il m'a donc paru qu'il convenait de préparer, en ce qui concerne la Mécanique, cette prochaine évolution."

Mr. William Allingham, who was connected with the marine department of the Meteorological Office of Great Britian for over forty years, died on January 24, 1919 in the sixty-eighth year of his age. He was a prolific writer. With D. W. Barker he was joint author of *Navigation: practical and theoretical* (London, 1896). He also edited the various editions of S. T. S. Lecky's *"Wrinkles" in practical navigation* beginning with the fifteenth in 1908.

On account of war conditions the publication of *Periodico di matematica per l'insegnamento secondario* and *Supplemento al Periodico di matematica* has been temporarily suspended. The last number of *Periodico* was volume 32, no. 6, October, 1917; of *Supplemento*, volume 20, no. 9, July, 1917.

The first number of a new scientific periodical, *El Progreso Cientifico*, has appeared recently at Saragossa under the direction of Professor Z. G. de Galdeano. It is to be a semi-annual review devoted to mathematics, physics, and chemistry, and containing papers dealing with fundamental questions, with criticism, and with scientific methodology, as well as with bibliographies, and matters pertaining to the teaching of science.

Dr. George Sarton, who has been connected with Harvard University and the Carnegie Institution of Washington for several years, expects to return to Belgium in June, but will be again at the Carnegie Institution in December. During his absence he will arrange for the resumption of the publication of *Isis*, the international quarterly (devoted to the history and philosophy of science) which he founded in 1913. The last part of volume 2, which was in the press in Brussels when the war broke out, will probable be issued in the autumn. The publication of volume 3 will take place soon after, perhaps in 1919, but at the latest in the early part of 1920.

Dr. Sarton has also announced in *Science* that "instead of publishing in four languages an effort will be made to use only French and English—chiefly, and perhaps exclusively, the latter. Articles written in other languages will be translated into English. The new *Isis* will only publish shorter articles. The longer and more monographic ones would be included in *Studies in the History and Method of Science* by Dr. Charles Singer of Exeter College, Oxford." The first volume of this work was issued by the Oxford University Press in 1917 and the second volume is now ready for the press. Dr. Singer is to share with Dr. Sarton the editorial responsibilities of the third and succeeding volumes of *Isis*. "Thus *Isis* and the *Studies* would be supplementary one to the other, and between them would provide suitable outlet for new work on the history and philosophy of science."

We list elsewhere a sketch of PETER LUDWIG MEJDELL SYLOW who died September 7, 1918, in the 86th year of his age. He had been professor of mathematics at the University of Christiania since 1898. Seven of his papers published before 1873 are listed in the *Royal Society Catalogue* and yet others in Poggendorff's *Biographisch-Literarisches Handwörterbuch*. Extensive biographical and bibliographical details are to be found in *Norsk Forfatter-Lexikon 1814–1880* of J. B. Halvorsen, volume 5 (Kristiania, 1901, pages 621–623). Sylow was co-editor of Abel's works and a member of the editorial committee of *Acta Mathematica*. A portrait and some biographical notes may be found in *Acta Mathematica 1882–1912. Table générale des tomes 1–35* (Upsala & Stockholm, 1913). The setting of some of his discoveries is indicated by H. F. BLICHFELDT in his *Finite Collineation Groups* (Chicago, 1917).

One of these discoveries is "Sylow's Theorem": "If p^α is the highest power of a prime p which divides the order of a group G, the sub-groups of G of order p^α form a single conjugate set and their number is congruent to unity, mod p."[1] The practical application of this theorem in determining the possible number of Sylow sub-groups of a given group requires considerable numerical computation before even tentative results can be obtained. This led to the publication by the Carnegie Institution of Washington, in 1916, of *A Sylow Factor Table of the First Twelve Thousand Numbers giving the possible number of Sylow sub-groups of a g oup of given order between the limits of 0 and 12,000*, by HENRY W. STAGER.

ARTICLES IN CURRENT PERIODICALS.

THE ALUMNI REGISTER, University of Pennsylvania, volume 21, no. 2, December, 1918: "Henry Brown Evans, M.E., Ph.D.," by H. P. Fry, 119–121.[2] [Mathematician and astronomer, appointed in 1918 dean of the Towne Scientific School.]

ANNALS OF MATHEMATICS, volume 20 (2), no. 1, September, 1918: "Functions of limited variation and Lebesgue integrals" by Goldie P. Horton, 1–8; "On the Teixeira construction of the unicursal cubic" by N. Altshiller, 9–12; "The functional equation $f[f(x)] = g(x)$" by G. A. Pfeiffer, 13–22; "The existence of the functions of the elliptic cylinder" by Mary F. Curtis, 23–34;

[1] Sylow, "Théorèmes sur les groupes de substitutions," *Mathematische Annalen*, vol. 5, 1872, pp. 584–594. For the particular statement of the theorem used above Stager refers to Burnside, *The theory of groups of finite order*, 2. ed., Cambridge, 1911, §§ 120 *et seq.*

[2] Unless stated to the contrary such numbers refer to pages.

"The gamma function in the integral calculus" by T. H. Gronwall, 35–76—No. 2, December: "The gamma function in the integral calculus" (concluded), 77–124; "Invariants which are functions of parameters of the transformation" by O. E. Glenn, 125–135; "A theorem on exhaustible sets connected with developments of positive real numbers" by H. Blumberg, 136–141; "Solution of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ and its application to some geometrical problems" by A. Pell, 142–148; "A general method of summation of divergent series" by L. L. Smail, 149–154.

Bulletin of the American Mathematical Society, volume 25, no. 3, December, 1918: "General aspects of the theory of summable series" by R. D. Carmichael, 97–131; "On the problem of the resistance integral" by T. Hayashi, 131–132; "Note on editions of von Staudt's Geometrie der Lage" by R. C. Archibald, 132–134; "Mathematical Periodicals" [Review of Union List of Mathematical Periodicals by D. E. Smith and Caroline E. Seely] by R. C. Archibald, 134–137.

The Journal of the Indian Mathematical Society, volume 10, no. 5, October, 1918: "A note in combinatory analysis" by R. Vythynathaswamy, 414–418; "Extension of M'Cay's theorem" by S. Narayanan, 419–422; "Euclid's Book on Divisions of Figures" [Review of R. C. Archibald's book], 423–428; "Note on Legendre's relation in elliptic functions" by K. B. Madhava, 429–430.

The Mathematics Teacher, volume 11, no. 1, September, 1918: "The New York state regents syllabus in intermediate algebra" by F. F. Decker, 1–8; "A geometric representation" (concluded) by E. D. Roe, 9–25; "The Reconstruction of the mathematical requirement" by G. W. Evans, 26–33; "A geometric illustration of limits" by C. E. Stromquist, 34–35; "Character-building content of arithmetic" by J. L. Green, 36–41—No. 2, December: "Why students fail in mathematics" by Helen A. Merrill, 45–56; "A solution of equations by standard curves" by R. C. Colwell, 57–60; "War problems in mathematics" by W. E. Breckenridge, 61–79; "Arithmetical errors made by high school pupils" by J. H. Minnick, 80–89; "Some relations connecting the sums of the coaxial minors of a circulant" by W. H. Metzler, 90–93; "Canada's challenge: In Flanders Field" by J. D. McCrae, 94; "America's Answer" by R. W. Lillard, 95.

Messenger of Mathematics, volume 48, no. 1, May, 1918: "The problem of the square pyramid" by G. N. Watson, 1–16.

The Monist, volume 28, October, 1918: "Leibnitz and Pascal" by K. I. Gerhardt with critical notes and a summary by J. M. Child and translations of Leibnitz's manuscripts alluded to by Gerhardt, 530–566; "The genesis of an electromagnetic field" by H. Bateman, 586–596; "Galileo and Newton" by P. E. B. Jourdain, 629–633.

Nature, volume 102, November 7, 1918: "Prof. Olaus Henrici, F.R.S." by M. J. M. Hill, 189–190—November 28: Review of R. C. Tolman's *The Theory of the Relativity of Motion* (Berkeley, 1917), 242–243—December 19: Review by S. Brodetsky of P. Frost's *An Elementary Treatise on Curve Tracing* (Fourth edition revised by R. J. T. Bell, London, Macmillan, 1918), 303–304.

The Nineteenth Century and After, volume 84, November, 1918: "On teaching mathematics" by Mrs. K. Lucas, 942–958.

Nyt Tidsskrift for Matematik, Copenhagen, volume 29, 1918, no. 1, June, *A*: En Sætning om Trekantens Røringscirkler" by J. Hjelmslev, 1–4; "Adgangseksamen til polyteknisk Læreanstalt, 1917" 5–8; "Landbohøjskolens Eksamensopgaver i Matematik, 1917" 8–10; "Studentexamen, 1917" 11–13; "Realskolexamen" 13–14. *B*: "Une formule exacte pour la détermination du nombre des nombres premiers au-dessous de x qui appartiennent à une classe de nombres donnée" by V. Brun, 1–8; "Über Mengen, die Elemente ihrer selbst sind" by H. Eklund, 8–28—No. 2, October, *A*: "En ny metod att diskutera den allmänna andragrads ekvationen med två variabler under förutsättning av snedvinkliga koordinataxlar" by G. Forsström, 25–39; "Ludwig Sylow, 1832–1918" 46–48. *B*: "Studier over en Afhandling af Gauss" by J. L. W. V. Jensen, 29–36; "Ueber uneigentliche Redeweisen in der Mengenlehre und über einen Aufsatz des Herrn H. Eklund" by T. Brodén, 36–43; "Skoleembedseksamen, Januar 1918," 43–50.

Proceedings of the American Academy of Arts and Sciences, volume 53, no. 5, March, 1918: "The dyadics which occur in a point space of three dimensions" by C. L. E. Moore and H. B. Phillips, 387–438—No. 8, July: "Rotations in hyperspace" by C. L. E. Moore, 649–694—No. 10, September: "Benjamin Osgood Peirce (1854–1914)" by E. H. Hall, 850–854.

Proceedings of the Edinburgh Mathematical Society, volume 36, part 1 (issued June, 1918): (1) "On the plane representation of the homaloidal surfaces which have a twisted cubic as multiple curve," 2–16, (2) "On a group of transformations connected with the 27 lines

of the non-singular cubic surface" by J. F. Tinto, 17–21; "Nicole's contribution to the foundations of the calculus of finite differences" by C. Tweedie, 22–39; "On a difference equation due to Stirling" by E. Pairman, 40–60—Part 2 (issued November): "The Brocard and Tucker circles of a quadrilateral" by F. G. W. Brown, 61–83; "The apolar locus of two tetrads of points on a conic" by W. P. Milne, 84–90; "Quaternion note on the theory of confocals" by C. G. Knott, 91–93; "An approximate value for the length of an arc of a suspended rope" by E. M. Horsburgh, 94–95; "Rolling loads: a new graphical method" by R. F. Muirhead, 96–102; "A formula for the solution of algebraic or transcendental equations" by E. T. Whittaker, 103–106; "On determinants whose elements are determinants" by E. T. Whittaker, 107–115.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES OF THE U. S. OF A., volume 4, 1918, no. 8, August: "Arithmetical Theory of certain Hurwitzian continued fractions" by D. N. Lehmer, 214–218; "On closed curves described by a spherical pendulum" by A. Emch, 218–221—No. 9, September: "On the α-holomorphisms of a group" by G. A. Miller, 293–294—No. 10, October: "Invariants and canonical forms" by E. J. Wilczynski, 300–305—No. 11, November: "On certain projective generalizations of metric theorems, and the curves of Darboux and Segre" by G. M. Green—No. 12, December: "On Jacobi's Extension of the continued fraction algorithm" by D. N. Lehmer, 360–364—Volume 5, no. 1, January, 1919: "A theorem on power series with an application to conformal mapping" by T. H. Gronwall, 22–24.

REVISTA DE MATEMATICAS, Buenos Aires, volume 2, no. 6, January–February, 1918: "La ecuación de las trayectorias ortogonales de las superficies equipotenciales newtonianas correspondientes a dos o más centros alineados y la construcción de las mismas" by B. I. Baidaff, 161–164; "Cuestiones de matemáticas elementales; relacionadas con la teoria de los grupos y con los principios del cálculo diferencial" (continued) by J. Duclout, 164–171; "Ejemplos de integración inmediata" (continued) by E. Rebuelto, 172–175; notas, bibliografia, problemas, ejercicios, tabla de las materias, 176–192.

SCIENCE, volume 48 (2), November 29, 1918: "Maxime Bôcher" [Minute on his life and services placed upon the records of the faculty of arts and sciences, Harvard University, at the meeting of October 22, 1918], 534–535—December 6: "Means for the scientific development of mathematics teachers" by G. A. Miller, 553–560; "A Greek tract on indivisible lines" by F. Cajori, 577–578—December 20: Reviews by C. L. Poor of Arrhenius's *The Destinies of the Stars* and of Hastings's *Modern Navigation*, 621–622—December 27: "International organization of Science" by G. A. Miller, 649–650—Volume 49, January 10, 1919: "Foundations of mechanics" by P. J. Fox, 44—February 14: "Edward Charles Pickering" by H. N. Russell, 151–155; "The publication of Isis" by G. Sarton, 170–171. [Quotations from Miller's article of December 6: ". . . I believe that if a man would secure a thorough knowledge of certain nine mathematical books beyond a first course in elementary calculus he would be much better informed than the average candidate for the Ph.D. degree. . . . The nine mathematical books whose mastery, together with a fair amount of general mathematical reading, and a development of some of the thoughts contained in these books, would make us an ornament unto our profession could be selected with considerable latitude. As one such selection the following may be noted: WEBER, *Lehrbuch der Algebra*, 3 volumes; GOURSAT, *Cours d'analyse mathématique*, 3 volumes; VEBLEN and YOUNG, *Projective Geometry*, 2 volumes—the second by Veblen alone; EISENHART, *Differential Geometry*, one volume. Those who do not read German might substitute for the three volumes of Weber's algebra the following: BÔCHER, *Introduction to Higher Algebra*; MILLER, BLICHFELDT, and DICKSON, *Finite Groups*; REID, *Theory of Algebraic Numbers*."]

THE TEXAS MATHEMATICS TEACHERS' BULLETIN, volume 4, no. 1, November, 1918: "Some observations on course of study, and mathematics in particular" by J. M. Bledsoe, 6–22; "A mathematical soothsayer" by P. M. Batchelder, 23–24; "Orientation for heavy (coast) artillery" by E. J. Oglesby, 25–29; "Progressive teaching of mathematics" (from *School Science and Mathematics*, May, 1918) by G. W. Myers, 30–40; "The straight edge," 41.

THE TÔHOKU MATHEMATICAL JOURNAL, volume 13, no. 4, June, 1918: "On irreducible equations admitting roots of the form $a + \rho e^{i\theta}$, a and ρ both rational" by A. Kempner, 253–265; "Note on Dr. Muir's paper on 'A Theorem including Cayley's on zero-axial skew determinants of even order'" by W. H. Metzler, 266–268; "Sur une propriété de la courbure de certaines courbes associées au triangle" by R. Goormaghtigh, 269–273; "On the null-system" by Y. Okada, 274–289; "Binary forms and duality" by K. Ogura, 290–295; "On the algebraic correspondence" by T. Kubota and S. Kakeya, 296–299; "Un Théorème sur les continus" by W. Sierpinski, 300–303; "Repeated solutions of a certain class of linear functional equations" by R. D. Carmichael, 304–

313; "Group-theory proof of two elementary theorems in number theory" by G. A. Miller, 314–315—Volume 14, nos. 1–2, August, 1918; "A determinantal theorem and Clifford's theorem on n lines" by T. Hayashi and K. Shibata, 1–10; "A construction-problem in elementary projective geometry" by T. Hayashi, 11–19; "Ueber die Schwerpunkte der konvexen geschlossenen Kurven und Flächen" by T. Kubota, 20–27; "Theory of the point-line convex (1, 1) in space, I" by K. Ogura, 28–63; "Theorems on convergent integrals" by T. Kojima, 64–79; "On the mean center of points on an algebraic curve" by K. Yanagihara, 80–89; "On the mean center of the contact points of tangent planes to an algebraic surface" by K. Shibata, 90–97; "Ueber die Konstruktionsaufgaben dritten und vierten Grades" by T. Kubota, 104–108; "Bemerkung zur Theorie der Approximation der irrationalen Zahlen durch rationale Zahlen" by M. Fujiwara, 109–115; "The stability of the parachute" by S. Brodetsky, 116–123; "A generalized Pascal theorem on a space cubic" by K. Ogura, 124–126; "On integral inequalities between two systems of orthogonal functions" by K. Ogura, 152–154; "Determination of the central forces acting on a particle whose equations of motion possess an integral quadratic in the velocities" by K. Ogura, 155–160.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 19, no. 4, October, 1918; "Spiral minimal surfaces" by J. K. Whittemore, 315–330; "On the group of isomorphisms of a certain extension of an abelian group" by L. C. Mathewson, 331–340; "Concerning the zeros of the solutions of certain differential equations" by W. B. Fite, 341–352; "Differentiation with respect to a function of limited variation" by P. J. Daniell, 353–362; "Linear integro-differential equations with a boundary condition" by M. F. Fu, 363–407; "On scalar and vector covariants of linear algebras" by Olive C. Hazlett, 408–420.

UNIVERSITY BULLETIN, Louisiana State University, volume 10, new series, no. 8, August, 1918: "An appreciation of James W. Nicholson"[1] by S. T. Sanders, 1–31. [Subheadings are: "A truth lover and truth seeker," "The mathematician," "The teacher," "Typical mathematical product," "Nicholson's method," and "Nicholson the philosopher." Nicholson was president of the university and head of the department of mathematics from 1883 to the time of his death in March, 1917.]

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT ALLER SCHULGATTUNGEN, volume 49, 1918, Heft 1, January: "Vermessungskunde in Trigonometrieunterricht" by H. Schuhmacher, 1–19 (see also 186)—Heft 2, February: "Eine Ergänzung der Archimedischen Kreismessung" by H. Dörrie, 41–45; "Ueber die Flächenwinkel einer dreiseitigen Ecke" by R. Sturm, 45–46; "Ueber Krümmungen verschiedener Ordnung" by R. Mehmke, 47–49; "Elementare Theorie der ebenen Sonnenuhren nebst einigen speziellen Bemerkungen zur Gnomonik der Araber"49–57; "Die neue preussische Prüfungsordnung für das Lehramt an höheren Schulen" by W. Lietzmann, 58–61—Heft 3, April: "Die Anfänge der analytischen Raumgeometrie" by H. Wieleitner, 73–79; "Elementargeometrische Behandlung der Dupinschen Zyklide" by K. Kommerell, 79–95—Doppel-Heft 4–5, June 13: "Ueber harmonische Kegelschnitte" by H. Pfaff, 113–127; "Ueber Rechenmaschinen und Rechenunterricht (ein Beitrag zu einer Reform der Methodik und Schematik auf kinematischer Grundlage)," 127–139; "Ergänzende Zusätze zu der Arbeit von Herrn Haentzschel 'Eine von Newton gestellte Aufgabe über Sehnenvierecke' (46. Jahrgang, S. 190–194, 1915)" by E. Lampe, 139–144; "Bemerkung zu den vorstehenden 'Zusätzen'" by E. Haentzschel, 144–145; "Das Stellenwertsystem bei den Maya und bei den Indern" by H. Wieleitner; "Berechnung rechtwinkliger Dreiecke bei den Akkadern um 2000 v. Chr." by W. Lietzmann 148–149—Heft 6, June 20; "Ueber die Rentabilität der auslösbaren Schatzanweisungen der 8. Kriegsanleihe und ihre Ermittlung" by P. Lötzbeyer, 161–164; "Eine für Ellipse und Hyperbel gleichlautende Achsenkonstruktion" by F. Redl, 165–170; "Berechnung der Einmal-Prämie für eine 'unterjährige' Leibrente" by K. Wolletz, 170–174; "Zur stetigen Teilung und zum Fünfeck" by W. Weber, 174–178—Heft 7, July: "Schulnomogramme" by P. Luckey, 193–203; "Ueber den Zusammenhang der Heronischen Inhaltsformel mit einigen Gleichungen der Kegelschnitte" by C. Ibrügger, 204–207; "Tangenssatz, Mollweidesche Formeln, Kosinussatz, acht Additionssätze an einer Figur" by W. Weber, 212–213; "Die Winkelhalbierenden des Sehnenvierecks" by C. Stengel, 213–215.

THE YALE ALUMNI WEEKLY, volume 28, no. 13, December 13, 1918: "Mathematics for freshmen and sophomores, how to make the subject more useful to students in a reorganized curriculum" by E. W. Brown, 310–311.

[1] He was the author of "A simple solution of the Diophantine equation $U^3 = V^3 + X^3 + Y^3$" in this MONTHLY, September, 1915, vol. 22, pp. 224–225.

AMERICAN DOCTORAL DISSERTATIONS.

The fifth *List of American Doctoral Dissertations* recently published by the Library of Congress[1] contains the following 22 titles in mathematics:

SUZAN R. BENEDICT, *A comparative study of the early treatises introducing into Europe the Hindu art of reckoning* . . . [Concord, N. H., 1916.] 6 + 126 pp. (Univ. of Michigan, 1914.)

R. W. BURGESS, 1887–, *The uniform motion of a sphere through a viscous liquid.* [Reprinted from *Amer. Journ. Math.*, Baltimore, 1916.] Pp. 81–96. (Cornell Univ., 1914.)

H. C. GOSSARD, 1884–, *On a special elliptic ruled surface of the ninth order* . . . [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 431–445. (Johns Hopkins Univ., 1914.)

MARY A. GRIGGS, *The surface tension of mixed liquids.* New York, 1916. 25 pp. (Columbia Univ., 1917.)

J. O. HASSLER, 1884–, *Plane nets periodic of period 3 under the Laplacian transformation.* [Reprinted from *Rendiconti del Circolo Mat. di Palermo*, 1916.] 22 pp. (Univ. of Chicago, 1915.)

OLIVE C. HAZLETT, *On the classification and invariantive characterization of nilpotent algebras.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 109–138. (Univ. of Chicago, 1915.)

F. T. H'DOUBLER, *A study of certain functional equations for the θ-functions*, by E. B. Van Vleck and F. T. H'Doubler. [Reprinted from *Trans. Amer. Math. Soc.*, Lancaster, Pa., 1916.] Pp. 9–49. (Univ. of Wisconsin, 1910.)

G. JOHNSON, 1872–, *The arithmetical philosophy of Nicomachus of Gerasa.* Lancaster, Pa., 1916. 3 + 49 pp. (Univ. of Pennsylvania, 1911.)

H. R. KINGSTON, 1886–, *Metric properties of nets of plane curves.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 407–430. (Univ. of Chicago, 1914.)

J. R. KLINE, 1891–, *Double elliptic geometry in terms of point and order alone.* [Reprinted from *Annals of Math.*, Lancaster, Pa., 1916.] Pp. 31–44. (Univ. of Pennsylvania, 1916.)

F. J. MCMACKIN, 1888–, *Some theorems in the theory of summable divergent series* . . . Lancaster, Pa., 1916. 23 pp. (Columbia Univ., 1916.)

R. W. MARRIOTT, 1882–, *Determination of the order of the groups of isomorphisms of the groups of order p^4, where p is a prime.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 139–154. (Univ. of Pennsylvania, 1911.)

L. C. MATHEWSON, *Theorems on the groups of isomorphisms of certain groups.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 19–44. (Univ. of Illinois, 1914.)

BESSIE I. MILLER, *A new canonical form of the elliptic integral.* [Reprinted from *Trans. Amer. Math. Soc.*, 1916.] Pp. 259–283. (Johns Hopkins Univ., 1914.)

W. L. MISER, 1886–, *On multiform solutions of linear differential equations having elliptic function coefficients.* [Reprinted from *Trans. Amer. Math. Soc.*, 1916.] Pp. 109–130. (Univ. of Chicago, 1913.)

E. J. MOULTON, *On figures of equilibrium of a rotating compressible fluid mass; certain negative results.* [Reprinted from *Trans. Amer. Math. Soc.*, 1916.] Pp. 100–108. (Univ. of Chicago, 1913.)

A. L. NELSON, 1891–, *Plane nets with equal invariants.* [Reprinted from *Rendiconti del Circolo Mat. di Palermo*, 1916.] 25 pp. (Univ. of Chicago, 1915.)

D. M. SMITH, 1884–, *Jacobi's condition for the problem of Lagrange in the calculus of variations.* [Reprinted from *Trans. Amer. Math. Soc.*, 1916.] Pp. 459–475. (Univ. of Chicago, 1916.)

L. A. H. WARREN, 1879–, *A class of asymptotic orbits in the problem of three bodies.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 221–247. (Univ. of Chicago, 1913.)

C. WOODY, 1884–, *Measurement of some achievements in arithmetic.* New York, Teachers College, 1916. 3 + 63 pp. (Columbia Univ., 1916.)

C. H. YEATON, 1886–, *Surfaces characterized by certain special properties of their directrix congruences.* [Reprinted from *Annali di mat.*, Milano, 1916.] 33 pp. (Univ. of Chicago, 1915.)

MABEL M. YOUNG, *Dupin's cyclide as a self-dual surface.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 267–286. (Johns Hopkins Univ., 1914.)

[1] *A List of American Doctoral Dissertations printed in 1916.* Washington, Government Printing Office, 1918.

PROBLEMS AND SOLUTIONS.

Edited by B. F. FINKEL and OTTO DUNKEL.

Send all communications about problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[Errata: *To the number of each problem for solution in the issue of February, 1919, 4 should be added*].

2757. Proposed by ERNEST P. LANE, Rice Institute, Houston, Texas.

Integrate by quadrature the differential equation

$$\frac{d^2y}{dx^2} - 3y\frac{dy}{dx} + y^3 = 0.$$

2758. Proposed by L. RICHARDSON, Vancouver, B. C., Canada.

Prove that, if r be a positive integer,

$$\int_0^{\pi/2} \frac{\sin(2r+1)\psi}{\sin\psi}\, d\psi = \frac{\pi}{2}$$

and

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin\psi}\, d\psi = 2\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{r-1}}{2r-1}\right\}.$$

2759. Proposed by J. L. RILEY, Stephenville, Texas.

Solve the simultaneous functional equations

$$\Phi(x+y) = \Phi(x) + \frac{\Phi(y)\cdot\psi(x)}{1 - \Phi(x)\Phi(y)},$$

$$\psi(x+y) = \frac{\psi(x)\psi(y)}{1 - \Phi(x)\Phi(y)}.$$

2760. Proposed by CHARLES N. SCHMALL, New York City.

In an arithmetical progression, if s_n be the sum of the first n terms, s_{2n} the sum of the first $2n$ terms, and s_{3n} the sum of the first $3n$ terms of the same series, prove that $s_{2n} - s_n = \frac{1}{3}s_{3n}$.

2761. Proposed by W. W. DENTON, Ann Arbor, Michigan.

Find the length of the side of an equilateral triangle whose vertices are at given distances a, b, c, from a given point.

SOLUTIONS OF PROBLEMS.

411 (Calculus) [June, 1916]. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

Prove that the volume bounded by the surface $f(x, y, z) = 0$ is $1/3\iint (z - x(\partial z/\partial x) - y(\partial z/\partial y))\, dxdy$ integrated over the area determined by projecting the surface on the xy-plane.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

The above problem is not stated accurately for all cases. The integral should be taken over the entire surface,—that is we have a surface integral, where the element of surface, dS, is replaced by its projection on the xy-plane in each element of the sum. This is not the same as integrating over the area described above.

Let us assume that the above surface is a closed, finite one with no double points or singularities. Then the direction cosines of the normal at any point are proportional to $-\partial z/\partial x$, $-\partial z/\partial y$, 1. If we denote the direction cosines by α, β, γ, and remember that $\gamma\cdot dS\cdot = dxdy$, the given integral becomes $1/3\iint(\gamma z + \alpha x + \beta y)dS$ taken over the surface. By a special case of Green's Theorem (B. O. Peirce, *Short Table of Integrals*, formula 883) this becomes $1/3\iiint(1+1+1)dxdydz$, taken throughout the volume inclosed by the surface, which is clearly the volume required.

That the above formula does not give the volume bounded by $f(x, y, z) = 0$, the xy-plane and a cylinder whose elements are parallel to the z-axis, may be readily seen by applying it to the plane $z = c$, in which case it gives a result one-third as large as the correct result.

Also solved by the Proposer.

239 (Number Theory) [March, 1916]. Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.

Give a general method for determining the solution in integers of the equation

$$x^r - 10xy - (n + 1) + y = 0,$$

where n and r are positive integers.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Solving for y, $y = \dfrac{x^r - (n + 1)}{10x - 1}$, which must be an integer. Since the denominator is prime to 10 and the numerator integral, y will be an integer if $\dfrac{10^r x^r - (n + 1)\cdot 10^r}{10x - 1}$ is. Dividing algebraically the remainder is $1 - (n + 1)\cdot 10^r$. If then $\dfrac{10^r(n + 1) - 1}{10x - 1}$ is an integer, so is y. Hence the general process (perhaps not that desired) is the following: form $10^r(n + 1) - 1$ and factor it. Equate $10x - 1$ to any factor whose last digit is 9, and we have an integral solution.

261 (Number Theory) [March, 1917]. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that for any positive integer n (excluding powers of 2) positive integers $a_1, a_2, a_3, \cdots, a_k$ which are less than $n/2$ can be chosen in such a way that

$$2^k \cos(a_1\pi/n) \cos(a_2\pi/n) \cos(a_3\pi/n) \cdots \cos(a_k\pi/n) = 1.$$

SOLUTION BY C. F. GUMMER, Queen's University.

Since n is not a power of 2, it is of the form $(2k + 1)l$. The equation

$$\cos(2k + 1)x - \cos(2k + 1)\alpha = 0$$

has $2k + 1$ distinct roots in $\cos x$, when $\cos(2k + 1)\alpha \neq 1$, the roots being

$$\cos\left(\alpha + \frac{2i\pi}{2k + 1}\right), \qquad i = 0, 1, \cdots, 2k.$$

Also $\cos(2k + 1)x = 2^{2k}\cos^{2k+1}x - \cdots$, the absolute term being zero. Hence,

$$2^{2k} \prod_{i=0}^{2k} \cos\left(\alpha + \frac{2i\pi}{2k + 1}\right) = \cos(2k + 1)\alpha.$$

By taking the limit of each side when $\alpha \to 0$,

$$2^{2k} \prod_{i=0}^{2k} \cos\frac{2i\pi}{2k + 1} = 1,$$

that is,

$$\left\{2^k \prod_{i=0}^{k} \cos\frac{2i\pi}{2k + 1}\right\}^2 = 1,$$

or

$$2^k \prod_{i=0}^{k} \cos\frac{2i\pi}{2k + 1} = \pm 1.$$

By taking $j = 2i$, when $i \leqq k/2$ and $j = 2k + 1 - 2i$ when $i > k/2$, we get

$$2^k \prod_{j=1}^{k} \cos\frac{j\pi}{2k + 1} = +1,$$

which takes the required form if $a_j = jl$.

2665, 2670 [Jan., Feb., Sept., and Dec., 1918]. **Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.**

A telegraph wire which weighs 1/10 of a pound per yard is stretched between poles on a level ground so that the greatest dip of the wire is 3 feet. Find approximately the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

III. Solution by J. B. Reynolds, Lehigh University.

Choosing any small portion of the wire of length as resolving tangentially and normally to the curve of equilibrium and passing to differentials we get the fundamental equations

$$T\frac{d\varphi}{ds} = w\cos\varphi, \qquad \frac{dT}{ds} = w\sin\varphi \tag{1}$$

in which T is the tension in the wire, w the weight per foot, φ the angle the tangent makes with the horizontal and s the length of the curve measured from the lowest point.

Dividing and integrating, we find

$$T = T_0\sec\varphi \tag{2}$$

T_0 being tension at lowest point. Putting this value of T in the second equation above we find

$$ws = T_0\tan\varphi. \tag{3}$$

Now since $\tan\psi = dy/dx$ and $ds/dx = \sqrt{1 + (dy/dx)^2}$ we may, from (3), arrive at the equation

$$y = \frac{T_0}{2w}\{e^{wx/T_0} + e^{-wx/T_0}\} - \frac{T_0}{w}, \tag{4}$$

y being measured vertically from the lowest point of the curve, which is taken as the origin. Likewise, from these equations we may find

$$T_0 = \frac{w}{8d}(l^2 - 4d^2), \tag{5}$$

where l is the length of wire between poles and d is the greatest dip, and

$$l = h\left\{1 + \frac{8}{3}\left(\frac{d}{h}\right)^2 - \frac{224}{45}\left(\frac{d}{h}\right)^4 \cdots\right\}, \tag{6}$$

in which h is the horizontal distance between poles. For this problem, $T_0 = 140$, $w = 1/30$; so, by (4),

$$y = 2100\{e^{x/4200} + e^{-x/4200}\} - 4200.$$

Expanding to x^2, we find, when $y = 3$, $x^2 = 25200$; whence $x = 159$ ft., $2x = 318$ ft., as a good approximation of the distance between poles.

If we use equation (5), we have $l^2 = 100836$; whence $l = 317$ ft.—a nearer value. Finally by using (6) we get $h = 316 +$ ft.

2679 [Feb., 1918]. **Proposed by J. W. LASLEY, JR., University of North Carolina.**

Show that the perpendicular from any point on a circle to any chord of the circle is a mean proportional to the perpendiculars from that point to the tangents at the ends of the chord.

Solution by C. E. Githens, Wheeling, W. Va.

Let CD be the perpendicular from the point C to any chord AB of the given circle, and let EC and CH be the perpendiculars from C to the tangents to the circle at the points A and B respectively. To prove $EC \times HC = CD^2$. The quadrilaterals $AECD$ and $BHCD$ are similar since the angles of the one are respectively equal to the angles of the other, that is, angle EAD = angle DBH, angle CDB = angle ADC = angle AEC = angle CHB, being right angles. Hence, the corresponding sides are proportional; that is, $EC : CD = CD : CH$, or $EC \times CH = CD^2$.

Also solved by C. A. Barnhart, Paul Capron, L. E. Mensenkamp, Roger A. Johnson, I. J. Fajans, Samuel Cohen, O. E. Simonsen, I. Millenky,

J. V. FAZIO, S. WEINBERGER, N. P. PANDYA, H. L. OLSON, ELIJAH SWIFT, W. L. LORD, J. W. LASLEY, T. R. THOMSON, ABRAHAM PLATMAN, H. GLADSTONE, PETER PUMO, ISADORE ROSE, R. M. MATHEWS, J. L. RILEY.

2680 [March, 1918]. **Proposed by C. C. YEN, Tangshan, North China.**

The diagonals of a maximum parallelogram inscribed in an ellipse are conjugate diameters of the ellipse. (From Joseph Edward's *Elementary Treatise on Differential Calculus.*)

SOLUTION BY L. E. MENSENKAMP, Freeport, Illinois.

Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$. Now, it is easily shown that a quadrilateral inscribed in an ellipse is a parallelogram when, and only when, the opposite vertices are symmetrical with respect to the origin. In other words, the diagonals of an inscribed parallelogram pass through the center of the ellipse. Let us denote one vertex of the parallelogram (which for convenience we may assume to be in the first quadrant) by $P_1 = (x_1, y_1)$. Let an adjacent vertex be $P_2 = (x_2, y_2)$. Then, the vertex opposite P_1 will be $P_3 = (-x_1, -y_1)$, and that opposite P_2 will be $P_4 = (-x_2, -y_2)$.

We may call P_2P_3 the base of the parallelogram. Its equation may be written

$$(y_1 + y_2)x - (x_1 + x_2)y - (x_1 + x_2)y_1 + (y_1 + y_2)x_1 = 0.$$

The equation in this form enables us to apply the usual formula for the distance from a point to a line to obtain the distance from P_1 to the line P_2P_3, which is the altitude of the parallelogram. This expression for the altitude, after some reduction, becomes

$$D = \frac{2(x_1y_2 - x_2y_1)}{\sqrt{(y_1 + y_2)^2 + (x_1 + x_2)^2}}. \tag{1}$$

Multiplying D by the length of the base P_2P_3, we find the area of the parallelogram to be $A = 2(x_1y_2 - x_2y_1)$. Making use of the fact that these points lie on the ellipse, and assuming for the moment that P_2 lies above the X-axis, the area becomes

$$A = \frac{2b}{b}(x_1\sqrt{a^2 - x_2^2} - x_2\sqrt{a^2 - x_1^2}).$$

The condition for a maximum is that $\partial A/\partial x_1 = 0$, and $\partial A/\partial x_2 = 0$. Both of these conditions lead to the same equation, namely,

$$x_1x_2 = -\sqrt{a^2 - x_1^2}\sqrt{a^2 - x_2^2}. \tag{2}$$

Therefore,

$$\frac{y_1y_2}{x_1x_2} = \frac{y_1y_2}{-\sqrt{a^2 - x_1^2}\sqrt{a^2 - x_2^2}} = -\frac{b^2}{a^2}.$$

The last member of this equation follows from the elimination of y_1 and y_2 by means of the equation for the ellipse. Since y_1/x_1 and y_2/x_2 represent the slopes of the diagonals of the parallelogram, and since their product equals $-b^2/a^2$, it is seen (Bôcher, *Plane Analytic Geometry*, p. 153) that the diagonals are conjugate diameters of the ellipse.

If P_2 is assumed to be below the X-axis, we must remember to use

$$y_2 = -\frac{b}{a}\sqrt{a^2 - x_2^2},$$

but the final result is the same. It is evident that the equation (2) gives a maximum, and not a minimum, parallelogram; for, assuming P_2 temporarily fixed while P_1 varies, we see that the area is zero when P_1 coincides with either P_2 or P_4.

Also solved by J. B. REYNOLDS.

2681 [March, 1918]. Proposed by PHILIP FRANKLIN, College of the City of New York.

Given n letters of one kind and $n - 1$ letters of another kind, in how many ways can they be arranged so that, moving along the arrangement from one end to the other, the number of letters of the first kind passed over is greater than the number of the second kind at any instant?

SOLUTION BY C. F. GUMMER, Queen's University.

Let $f(p, q)$ be the number of ways of arranging p letters, some A and some B, so that, on going from left to right, we pass *always* more A's than B's, and *finally* q more A's than B's. Clearly $f(p, q) = 0$ when $p - q$ is odd and when $q < 1$.

Such an arrangement will end with an A in $f(p-1, q-1)$ cases, $(p > 1)$; and it will end with a B in $f(p-1, q+1)$ cases $(q \neq 0)$. Therefore,

$$f(p, q) = f(p-1, q-1) + f(p-1, q+1)$$

under the above restrictions.

To remove the restriction $q \neq 0$, let us define

$$g(p, q) = f(p, q), \ (q \geqq 0);$$
$$g(p, q) = -f(p, -q), \ (q < 0).$$

Then

$$g(p, q) = g(p-1, q-1) + g(p-1, q+1)$$

provided $p > 1$.

By successive application,

$$g(p, q) = g(p-2, q-2) + 2g(p-2, q) + g(p-2, q+2) = g(p-3, q-3)$$
$$+ 3g(p-3, q-1) + 3g(p-3, q+1) + g(p-3, q+3) = \cdots = g(1, q-p+1)$$
$$+ \binom{p-1}{1} g(1, q-p+3) + \binom{p-1}{2} g(1, q-p+5) + \cdots.$$

Now $g(1, q) = 0$, except that $g(1, 1) = 1$ and $g(1, -1) = -1$.

Hence, if $p - q$ is even,

$$g(p, q) = -\binom{p-1}{(p-q-2)/2} + \binom{p-1}{(p-q)/2}$$
$$= \frac{2q}{p+q}\binom{p-1}{(p-q)/2}.$$

For the particular problem, $p = 2n - 1$, $q = 1$, and

$$f(p, q) = g(p, q) = \frac{1}{n}\binom{2n-2}{n-1} = \frac{\lfloor 2n-2}{\lfloor n \, \lfloor n-1}.$$

2683 [March, 1918]. Proposed by J. R. HITT, Mississippi College.

The height of a frustum of a cone is h, the radii of the upper and lower circular bases are a and b, respectively. Deduce the formula for finding the center of gravity of the frustum.

SOLUTION BY H. C. GOSSARD, U. S. Naval Academy.

Since the center of gravity is, obviously, on the axis of the frustum, let y be the required distance of the center of gravity above the lower base. Complete the cone and let x be the altitude of the upper cone. From similar triangles, $x : x + h = a : b$; whence, by division, $x : h = a : b - a$, or $x = ah/(b-a)$. The altitude of the entire cone is $x + h = bh/(b-a)$.

The volume of the whole cone is $\frac{\pi}{3}\frac{hb^3}{(b-a)} = V_1$ and the volume of the upper cone is $\frac{\pi}{3}\frac{ha^3}{(b-a)} = V_2$.

Hence, the volume of the frustum is $\frac{\pi}{3}\frac{hb^3}{(b-a)} - \frac{\pi}{3}\frac{ha^3}{(b-a)} = \frac{\pi}{3}h(a^2 + b^2 + ab) = V_3$.

Remembering that the center of gravity of a cone is on its axis of symmetry and 1/4 of the distance from the center of gravity of the base to the vertex measured from the base, we have, on taking moments about any diameter of the base, $V_3 y = \frac{1}{4}bh/b - aV_1 - \frac{1}{4}ah/b - aV_2$. Substituting the values of V_1, V_2, and V_3 and solving for $\bar{y}$ and reducing to simplest form, we have

$$y = \frac{h}{4}\left(\frac{b^2 + 2ab + 3a^2}{b^2 + ab + a^2}\right).$$

Also solved by H. L. OLSON, ALBERT N. NAUER, JOSEPH B. REYNOLDS, C. E. FLANAGAN, G. PAASWELL, and HERBERT N. CARLETON.

2684 [March, 1918]. Proposed by B. J. BROWN, Kansas City, Missouri.

Find the locus of the center of a conic passing through four fixed points.

I. SOLUTION BY H. D. THOMPSON, Princeton University.

This is an exercise given in books on coördinate geometry.

Take the x-axis through two of the four points, and the y-axis, oblique, through the other two. Let $1/l$ and $1/l'$ be the abscissas of the two points on the x-axis and $1/m$ and $1/m'$, the ordinates of the two points on the y-axis. Then $lx + my - 1 = 0$ and $l'x + m'y - 1 = 0$ form another pair of lines, containing the four points in pairs. All conics through the four points are given by $\lambda xy + (lx + my - 1)(l'x + m'y - 1) = 0$, when λ is the parameter of the system.

The center of a representative conic is given by $2ll'x + (lm' + l'm)y - (l + l') + \lambda y = 0$ and $(lm' + l'm)x + 2mm'y - (m + m') + \lambda x = 0$. Eliminating λ, the locus of the center is $2ll'x^2 - 2mm'y^2 - (l + l')x + (m + m')y = 0$, a conic through the origin. The center of the locus is $\{1/4(1/l' + 1/l), 1/4(1/m' + 1/m)\}$. As any pair of opposite sides of the complete quadrilateral with the original four points as vertices can be taken as the axes, the locus of the centers will pass through the three points of intersection of opposite sides of the complete quadrilateral.

The locus is an hyperbola when $ll'mm'$ is positive, that is, when the original four points may be taken as the vertices of a convex polygon; and it is in this case only that the original conic may be a parabola (two).

II. SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The equation of a conic passing through the four given points $\pm\alpha_1, \pm\beta_1, \pm\gamma_1$, trilinear coördinates being used, is of the form

$$l_1\alpha^2 + m_1\beta^2 + n_1\gamma^2 = 0 \tag{1}$$

with the condition

$$l_1\alpha_1^2 + m_1\beta_1^2 + n_1\gamma_1^2 = 0. \tag{2}$$

The coördinates of the center of (1) are given by

$$\frac{l_1\alpha}{a} = \frac{m_1\beta}{b} = \frac{n_1\gamma}{c}, \tag{3}$$

a, b, c, being the sides of the fundamental triangle.

Eliminating l_1, m_1, n_1 from (3) and (2), we have

$$\frac{a\alpha_1^2}{\alpha} + \frac{b\beta_1^2}{\beta} + \frac{c\gamma_1^2}{c} = 0, \tag{4}$$

the required locus.

This is the nine-point conic of the quadrilateral whose vertices are the four given points.

Also solved by PAUL CAPRON and ELIJAH SWIFT.

2685 [March, 1918]. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

A particle is describing an ellipse of eccentricity $\sqrt{2/3}$ as a central orbit about a focus when the attracting force suddenly becomes repulsive without changing its magnitude and the particle begins to describe an equilateral hyperbola; find where the change occurred and the angle that the major axis of the new orbit makes with that of the old orbit.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The focal equation of the ellipse is

$$p_1^2 = \frac{a_1^2(1 - e_1^2)r_1}{2a_1 - r_1} \tag{1}$$

and of the hyperbola,

$$p_2^2 = \frac{a_2^2(e_2^2 - 1)r_2}{2a_2 + r_2} \tag{2}$$

an r being a radius vector; e, eccentricity; a, semi-major axis; p, perpendicular from a focus upon the corresponding tangent. In the ellipse, $e_1^2 = 2/3$, and in the hyperbola, $e_2^2 = 2$.

For the central force in (1),

$$F_1 = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{a_1(1-e_1^2)} \cdot \frac{1}{r_1^2}, \tag{3}$$

h being the usual constant in the theory of central forces.

By condition, we may write

$$-\frac{h^2}{a_1(1-e_1^2)} \cdot \frac{1}{r_2^2} = \frac{h^2}{p_2^3}\frac{dp_2}{dr_2}. \tag{4}$$

Integrating,

$$\frac{1}{a_1(1-e_1^2)} \cdot \frac{1}{r_2} = -\frac{1}{2p_2^2} + C \tag{5}$$

Let $r_2 = c = r_1$ at the instant of change in the nature of the force, when also

$$p_2^2 = p_1^2 = \frac{1}{a_1^2(1-e_1^2)}\frac{2a_1 - c}{c} \tag{6}$$

by (1); then

$$C = \frac{4a_1 - c}{2a_1^2(1-e_1^2)c} \tag{7}$$

and (5) becomes

$$\frac{1}{p_2^2} = \frac{(4a_1 - c)r_2 - 2a_1c}{a_1^2c(1-e_1^2)r_2} \tag{8}$$

or,

$$p_2^2 = \frac{a_1^2c(1-e_1^2)}{4a_1 - c}r_2 \div \left\{ r_2 - \frac{2a_1c}{4a_1 - c} \right\}$$

which is plainly the hyperbola in (2).

Now, (9) and (2) are identical if

$$a_2^2(e_2^2 - 1) = \frac{a_1^2c(1-e_1^2)}{4a_1 - c} \tag{10}$$

and

$$2a_2 = -\frac{2a_1c}{4a_1 - c}. \tag{11}$$

Eliminating a_2, and solving for c, $c = a_1$ (12), showing that the nature of the force changes at an extremity of the minor axis of the ellipse.

Let α = the angular coördinate of this point, the position of the center of force being the pole, γ = the required angle of the problem, and l_1, l_2, the latera recta of the curves; then the focal polar equations of the curves are

$$\frac{l_1}{r} = 1 - e_1\cos\theta \quad (13), \qquad \frac{l_2}{r} = 1 - e_2\cos(\theta - \gamma) \quad (14).$$

The equations of the tangent to these at the common point α are

$$\frac{l_1}{r} = \cos(\theta - \alpha) - e_1\cos\theta \quad (15), \qquad \frac{l_2}{r} = \cos(\theta - \alpha) - e_2\cos(\theta - \gamma) \quad (16),$$

or, in cartesian coördinates,

$$(\cos\alpha - e_1)x + y\sin\alpha = l_1 \tag{17}$$

$$(\cos\alpha - e_2\cos\gamma)x + (\sin\alpha - e_2\sin\gamma)y = l_2. \tag{18}$$

These are identical if

$$\frac{\cos\alpha - e_1}{l_1} = \frac{\cos\alpha - e_2\cos\gamma}{l_2} \quad (19), \qquad \frac{\sin\alpha}{l_1} = \frac{\sin\alpha - e_2\sin\gamma}{l_2} \quad (20).$$

But from the ellipse, $\cos\alpha = e_1$, and (19) gives

$$\cos\gamma = \frac{e_1}{e_2} = \tfrac{1}{3}\sqrt{3}. \tag{2}$$

It may be added that the condition that (13) and (14) have the kind of contact consistent with the nature of the problem is

$$l_1^2(1 - e_2^2) + l_2^2(1 - e_1^2) - 2l_1l_2 + 2l_1l_2e_1e_2 \cos \gamma = 0. \qquad (22)$$

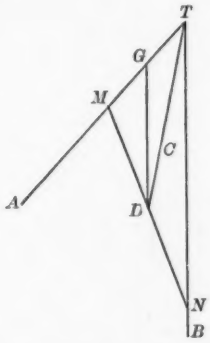
Also solved by HORACE L. OLSON.

2687 [March, 1918]. Proposed by N. P. PANDYA, Sojitra, India.

An ellipse intersects a parabola in A and B, and the tangents at A and B to the parabola meet at T. The center C of the ellipse lies within the space enclosed by the parabola and the tangents. Draw a third tangent to the parabola such that C may be the centroid of the triangle formed by the three tangents.

SOLUTION BY HORACE L. OLSON, Heidelberg University, Tiffin, Ohio.

This problem can be solved, if at all, without reference either to the ellipse or to the parabola. For this purpose let us alter the problem to read as follows: "Given two straight lines, TA and TB, intersecting at T, and a point C; draw a third line which shall form, with TA and TB, a triangle whose centroid shall be the point C." The problem, as thus stated, has a unique solution. If the third straight line is tangent to the parabola mentioned above, it is the solution of the original problem; if not, there is no solution. Draw the line TC, and extend it beyond C to D, so that $CD = \frac{1}{2}TC$. D will then be the mid-point of the third side of the required triangle. Through D, draw the line GD parallel to TB.



Lay off $GM = TG$. The line MDN, intersecting TB at N, will then be the required line; for, since the line GD is parallel to the side TN of the triangle TMN and bisects the side TM, it must also bisect the side MN. Since this demonstration is reversible, MN is the only line-segment included between the lines TA and TB, and bisected at D. Hence, if MN is tangent to the given parabola, it is the solution of the original problem; if not, there is no solution.

Also solved analytically by WILLIAM HOOVER.

2691 [April, 1918]. Proposed by ROGER A. JOHNSON, Hamline University.

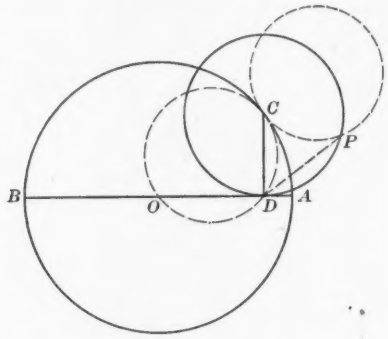
Show by purely geometric methods, without the use of the calculus, that the envelope of the circles whose centers are on a fixed circle and which touch a fixed diameter of that circle is a two-arched epicycloid. Cf. problem 423 (calculus) [February and September, 1917].

SOLUTION BY THE PROPOSER.

First, let us make a few general remarks about the envelope of a system of circles. In general, the points where one of a system of curves meets the envelope of the system are the limiting positions of the points of intersection of that curve with a near-by one, as the latter comes into coincidence with the former. But two circles generally intersect in two points; the perpendicular bisector of the line connecting them is the line joining the centers of the circles. Hence,

If a system of circles has an envelope, then generally each circle of the system meets the envelope at two points; the chord connecting these points is perpendicular to the diameter to said circle which is tangent to the curve of centers.

In the problem before us, a part of the envelope is known; namely, the fixed diameter. Let AB be a diameter of the circle with center O, radius R; let CD be the perpendicular from a point C of the circumference on the diameter AB. Then the circle with center C, radius CD, is one of the system whose envelope is to be found; and our problem reduces to that of determining the locus of the point P, which is symmetrical to D with regard to the tangent to the given circle at C.



Consider the circle having OC as diameter. Since ODC is a right angle, it passes through D. Hence, the circle symmetrical to it, in other words, the circle whose diameter equals R and which touches the given circle externally at C, passes through P.

Further, arc PC = arc DC = arc CA, since the one radius is half the other, and the angle COD inscribed in the one arc is a central angle with regard to the other.

Hence, P is the locus of a point fixed on the circle PC, of radius 1/2 R, as this circle rolls without sliding around the circumference of the given circle. In other words, the locus is a two-arched epicycloid.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Illinois.

The April meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Friday and Saturday, April 4, 5. Besides the usual research papers there will be a Symposium on "Geometry of Numbers" on Friday afternoon. Two formal papers will be presented; one by Professor H. F. BLICHFELDT, of Leland Stanford University, and the other by Professor L. E. DICKSON, of the University of Chicago.

A school of navigation is being conducted this winter at Queen's University, Kingston, Canada. The course is for Canadian lake and river mariners. Instruction is given for master's and instructor's certificates, and for those who cannot attend the day courses classes are held at night. The University mathematics and physics departments furnish the lectures on these scientific subjects.

Colleges and universities offering summer courses in mathematics are requested to report an outline of them to the editor of Notes and News as soon as possible. The following reports have been received:

Brown University, June 30–August 30, is to offer four courses each with a session of five hours per week: Solid geometry by Professor R. C. ARCHIBALD; Topics in geometry by Professor Archibald; Elementary calculus by Professor C. H. CURRIER; Analytic geometry and calculus (for engineers) by Professor CURRIER.

University of California, June 30–August 9. The University will conduct two Summer Sessions, one at the State Normal School in Los Angeles, the other at the University in Berkeley. The courses offered are: At Los Angeles, Professor F. P. BRACKETT: Plane trigonometry, Introduction to calculus; Professor B. A. BERNSTEIN: Plane analytic geometry, Logic of mathematics (Teacher's course). At Berkeley, Professor PUTNAM: Graphic algebra, Analytic geometry; Dr. PAULINE SPERRY: Solid geometry, Plane trigonometry; Professor F. CAJORI: Teaching of secondary mathematics, History of mathematics. Professor F. MORLEY, of Johns Hopkins University, will give a course on special fields of Geometry, together with a seminary on topics covered in the lecture course.

University of Iowa, Summer Session, first term, June 16–July 26. Professor H. L. RIETZ: Teacher's Course; Professor E. W. CHITTENDEN: Analytic Geometry, Calculus, Reading and Research; Mr. R. E. GLEASON: College Algebra; Trigonometry; Differential Equations. Second term, July 28–Aug. 30. Professor E. W. CHITTENDEN: Analytic Geometry, Calculus, Reading and Research.

Ohio State University. The following courses will be offered by Professors R. D. BOHANNAN, KUHN and C. C. MORRIS: College algebra, Plane trigonometry, Analytic geometry (10 hours); Differential calculus (5 hours); Integral Calculus (5 hours); Differential equations (3 hours); Fundamental concepts of algebra and geometry (2 hours). A course for teachers and those intending teaching as a profession.

Dr. C. E. WILDER has accepted a position at Clark University.

Dr. W. E. MILNE, formerly of Bowdoin College but more recently a first lieutenant in the ordnance reserve corps, has been appointed assistant professor of mathematics in the University of Oregon.

Professor C. N. MILLS has resumed his teaching in South Dakota State College after having served in the National Army.

Professor J. B. SMITH, formerly professor of mathematics in Hampden-Sidney College, is serving with the rank of first lieutenant as chief of the statistical division, office of the director of purchase and storage, at Washington.

Mrs. E. R. BECKWITH, assistant professor of mathematics in the College for Women, Western Reserve University, was in charge of a round table discussion

on accounting, insurance, statistics, and drafting, at the fourth vocational conference for women held at Oberlin College on February 8.

Dr. W. H. GARNETT, Dean and Professor of Mathematics at Wesleyan College, Winchester, Ky., is spending the year in California to rest, after thirty-seven years of continuous service at Wesleyan.

At the University of Kentucky, Mr. W. W. ELLIOTT has been appointed Assistant in Mathematics and Mr. H. P. PETTITT holds a fellowship in mathematics.

Professor A. L. RHOTON, of the mathematics department at Georgetown, Ky., College, has resigned to accept the position of Associate Professor of Education at Pennsylvania State College.

Mr. WILL E. EDINGTON, who has been engaged in the study of exterior ballistics in the meteorological section, Division of Science and Research, Signal Corps, at Fort Monroe, has returned to the University of Illinois as assistant in mathematics and will continue his graduate work.

Professor E. R. HEDRICK, of the University of Missouri, Director of Mathematics in connection with the work of the Y. M. C. A. for the American soldiers in France, will probably be gone for a year or more. Professor J. W. YOUNG of Dartmouth College is engaged in executive work in connection with organizing this overseas school for the soldiers. His headquarters for this work are in New York City.

G. X. P. KOENIGS, professor of physical and experimental mechanics at the University of Paris, has been elected a member, in the section of mechanics, of the Academy of Sciences of the Institute of France, in place of the late Professor H. C. V. J. LÉAUTÉ.

Dr. F. W. REED has been appointed instructor in mathematics at the University of Illinois.

Mr. J. S. MIKESH has been appointed instructor in mathematics at Sheffield Scientific School, Yale University.

Mr. A. D. CAMPBELL and Dr. C. M. SMITH have been appointed instructors in mathematics at Cornell University.

CHARLES WOLF, the French astronomer, died July 4, 1918, in the ninety-first year of his age.

E. C. PICKERING, professor of practical astronomy and director of the Harvard College observatory since 1876, died February 3, 1919, in the seventy-third year of his age.

Miss Z. FERGUSON, instructor in mathematics at the University of Missouri for the S. A. T. C. term, returned to her previous position in St. Joseph, Mo., in January.

Dr. E. A. KIRCHER, for a time Benjamin Peirce instructor at Harvard, but more recently battery commander, battalion adjutant and acting battalion commander on the Western Front, has accepted a position in connection with the National City Bank, New York City.

The Royal Society Copley Medal for 1918 was awarded to Professor H. A. LORENTZ, of the University of Leiden, for his distinguished researches in mathematical physics.

Mr. P. A. FRALEIGH has returned from military service at the Aberdeen Proving Ground to resume his duties as instructor in mathematics at Cornell University.

Dr. PAUL CARUS, editor and manager of the *Open Court* and *The Monist* at La Salle, Illinois, died on February 11, 1919, in the sixty-seventh year of his age. He was a very prolific writer. Among his books are: *The Foundation of Mathematics*, 1908, and *The Principle of Relativity in the Light of the Philosophy of Science*, 1913. He wrote also some chapters in *Magic Squares and Cubes* by W. S. Andrews, 1908.

In the list (published in *The Times*, London, January 9) of promotions and appointments in connection with the Civil Division of the Order of the British Empire, for services in connection with the war, occur the names of Commander T. A. FERRIER, Mathematical Instrument Office, Calcutta, India; Officer A. W. CODD, chief cartographer, Hydrographic Department, Admiralty; and Officer J. W. FOORD-KELCEY, professor of mathematics and mechanics, Royal Military Academy.

In December, 1916, the Indian Mathematical Society held its "first conference" at Madras. The proceedings were opened by His Excellency the Governor of Madras and the conference lasted for three days. It was announced that the second conference would meet at Bombay, January 11–13, 1919.—Among the five members of the Society who were recent victims of Spanish influenza, was a frequent contributor to its *Journal*, Mr. R. J. POCOCK, director of the Nizamiah Observatory, Begumpet, who died in November, 1918.—Apart from India's mathematical activity in connection with this Society it will be recalled that A. R. Forsyth's *Lectures Introductory to the Theory of Functions of Two Complex Variables* were delivered at the University of Calcutta in 1913, and that C. E. Cullis's work on *Matrices and Determinoids*, of which two large volumes have already appeared, was the outcome of "Readership Lectures" at the University of Calcutta where Mr. CULLIS is Hardinge professor of mathematics.

The "Committee on the MAXIME BÔCHER MEMORIAL" appointed by the American Mathematical Society, with Professor T. S. FISKE as chairman, consists of 80 members at present. It is hoped that a fund of two or three thousand dollars may be procured. "Contributors are invited to express their views and wishes in regard to the use which should be made of the fund. One suggestion is that the income of the fund be used once in four years to send to the International Congress of Mathematicians one or more delegates chosen by the Council to represent the American Mathematical Society. Another suggestion is that the income of the fund be used from time to time to assist a promising student in the last year of his study for the doctorate, perhaps to send him to the University of Paris, and that the theses of all students thus assisted be suitably inscribed. Still another suggestion is that the income of the fund be used to assist in the publication of meritorious scientific works for which otherwise it might be difficult or impossible to secure a publisher. . . . Subscriptions should be addressed to the treasurer of the Committee, Professor J. H. Tanner, Cornell University, Ithaca, New York."

University of Wisconsin

Summer Session, 1919

June 30 to August 8 (Law School, June 23 to August 29)

320 Courses. 160 Instructors. Graduate and undergraduate work leading to the bachelor's and higher degrees. Letters and Science, Medicine, Engineering, Law, and Agriculture (including Home Economics).

Teachers' Courses in high-school subjects. Strong programs in all academic departments. Exceptional research facilities. Favorable Climate. Lakeside Advantages.

Mathematics: See list of courses in News Column of this issue.

One fee for all courses, $15, except Law (10 weeks) $25.

For further announcements address

Registrar, University, Madison, Wisconsin

The Summer Session of Cornell University

Opens July 5, 1919 **Closes August 15, 1919**

The Department of Mathematics offers Courses in the following Subjects

Solid Geometry, Advanced Algebra, Trigonometry, Analytic Geometry and Differential and Integral Calculus, Advanced Calculus, Differential Equations, Advanced Geometry, Analysis.

Members of the Department of Mathematics also offer to direct reading or research along various lines whether or not closely related to courses given. For further information address

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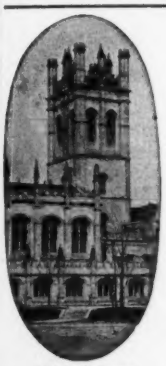

Mitchell Tower

The University of Chicago

Offers instruction during the Summer Quarter on the same basis as during the other quarters of the academic year.

The undergraduate colleges, the graduate schools, and the professional schools provide courses in **Arts, Literature, Science, Commerce and Administration, Law, Medicine, Education, and Divinity.** Instruction is given by regular members of the University staff which is augmented in the summer by appointment of professors and instructors from other institutions. For **Mathematics** courses see News Column.

Summer Quarter, 1919

1st Term June 16--July 23 2d Term July 24--Aug. 29

Detailed announcements will be sent upon application

THE UNIVERSITY OF CHICAGO Chicago, Ill.

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Ready in June

A book which covers the work usually given in college algebra and analytic geometry, and which makes use of the fundamentals of calculus. It is based on the principle of correlation which Dr. Moore, as President of the American Mathematical Society, advocated some years ago and which is gradually replacing the traditional method of teaching mathematics in "water-tight compartments," to borrow Dr. Moore's phrase.

Since the various subjects of mathematics are fundamentally related, college algebra and analytic geometry are, in this book, taught so as to be made helpful to each other. In this way the student gets a better understanding of the meaning and of the general methods of mathematics, and he is better able to appreciate functional correspondence.

The following table of contents will be of interest:

I. Location of a Point, Number-System.
II. The Straight Line, Linear Function.
III. Several Straight Lines. Simultaneous Linear Equations in Two Variables. Areas.
IV. Determinants.
V. Quadratic Function. Parabola.
VI. Rational Integral Functions of Degree Higher than the Second.
VII. Solution of Equations with Numerical Coefficients.
VIII. Algebraic Solution of the General Cubic and Biquadratic Equations.
IX. Limits.
X. Infinite Series.
XI. Partial Fractions.
XII. Permutations and Combinations.
XIII. The Circle.
XIV. Ellipse. Hyperbola. Parabola.
XV. Conic Sections. Transformation of Co-ordinates.
XVI. The General Equation of the Second Degree. Diameters.
XVII. Solid Analytic Geometry. Linear Functions.
XVIII. Solid Analytic Geometry. Quadratic Functions.

Although suitable for use in the freshman college year or in the junior college, this book will complete the four-year series in correlated mathematics for secondary schools, which has been worked out by E. R. BRESLICH and his staff of mathematics teachers in the University of Chicago High School, and therefore will bear the title:

Fourth-Year Mathematics for Secondary Schools

The price will be $1.25, postage extra. When placing your order, to be filled upon publication of the book, please be sure to indicate your June address *in addition to* your present address.

The University of Chicago Press

5876 Ellis Avenue **Chicago, Illinois**

FERMAT NUMBERS $F_n = 2^{2^n} + 1$.[1]

By R. D. CARMICHAEL, University of Illinois.

Introduction and Historical Remarks.

1. One of the most fascinating of the small separated regions of mathematics is that pertaining to the Fermat numbers[2]

$$F_n = 2^{2^n} + 1.$$

Fermat erred in the belief that every F_n is a prime; though he admitted that he had no proof. He challenged other mathematicians to furnish a demonstration of this beautiful proposition, adding that such a proof would probably give the key to penetrate all the mystery of prime numbers and would release his energies so that afterwards nothing could keep him back in these matters.

A most surprising geometric connection of the numbers F_n was brought to light when Gauss proved that a regular polygon of m sides can be constructed by ruler and compasses if m is a product of a power of 2 and distinct odd primes each of the form F_n and stated that the construction is impossible when m is not such a product.

2. For $n = 0, 1, 2, 3, 4$, the numbers F_n are 3, 5, 17, 257, 65537, respectively. It is easy to verify directly that these are primes. No other prime F_n is known as such, while for $n = 5, 6, 7, 8, 9, 11, 12, 18, 23, 36, 38, 73$ it has been shown that F_n is composite. A table of all the known factors of F_n is to be found in this MONTHLY, Vol. 21, 1914, p. 249, except for the required addition that F_{73} has the prime factor $2^{75}.5 + 1$. (See also § 16 of the present paper.) The factors actually listed in this table are all primes. The known factorization is complete for $n = 5, 6$, but is not known to be complete for greater values of n.

Euler was the first to prove the falsity of Fermat's conjecture that every F_n is prime; and this he did by pointing out that 641 is a factor of F_5. The American calculator, Zera Colburn, in his autobiography, records that while on exhibition in London, at the age of eight, he found "by the mere operation of his mind" that $F_5 = 641 \cdot 6700147$.

In view of the known facts about the factors of F_n, Fermat's question, whether $(2k)^{2^n} + 1$ is always a prime except when divisible by an F_n, is without further particular interest.

3. The principal problem so far studied in connection with the numbers F_n is that of their factorization. The most elementary and obvious method for finding the factors of a given number is to test it for divisibility by primes less than its square root; but if the given number is a large prime or has only large prime

[1] Presented to the American Mathematical Society (Southwesten section), November 30, 1918.

[2] A complete history of these numbers (with full references) is given in Dickson's *History of the Theory of Numbers*, Vol. I, 1919, Chap. XV. To this the reader is directed for further references to the literature.